# UNCERTAINTY PRINCIPLE - A STRONGER FORM

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 9.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Shankar's derivation of the general uncertainty principle relating the variances of two Hermitian operators actually gives a different result from that in Griffiths. To follow this post, you should first review the earlier post. To keep things consistent I'll use the original Griffiths notation up to equation 11, which is a summary of the earlier post.

Shankar's derivation is the same as Griffiths's up to equation (13) in the earlier post. To summarize, we have two operators $\hat{A}$ and $\hat{B}$ and calculate their variances as

$$\sigma_A^2 = \left\langle \Psi|(\hat{A}-\langle A\rangle)^2\Psi\right\rangle \tag{0.1}$$
$$= \left\langle \left(\hat{A}-\langle A\rangle\right)\Psi|\left(\hat{A}-\langle A\rangle\right)\Psi\right\rangle \tag{0.2}$$
$$\equiv \langle f|f\rangle \tag{0.3}$$

where the function $f$ is defined by this equation.

Similarly, for $\hat{B}$:

$$\sigma_B^2 = \left\langle \Psi|(\hat{B}-\langle B\rangle)^2\Psi\right\rangle \tag{0.4}$$
$$= \left\langle \left(\hat{B}-\langle B\rangle\right)\Psi|\left(\hat{B}-\langle B\rangle\right)\Psi\right\rangle \tag{0.5}$$
$$\equiv \langle g|g\rangle \tag{0.6}$$

We now invoke the Schwarz inequality to say

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \tag{0.7}$$
$$\geq |\langle f|g\rangle|^2 \tag{0.8}$$

At this point, Griffiths continues by saying that

$$\langle f|g\rangle|^2 \geq (\Im\langle f|g\rangle)^2 \tag{0.9}$$

That is, he throws away the real part of $\langle f|g\rangle$ to get another inequality. Shankar retains the full complex number and thus states that

$$|\langle f|g\rangle|^2 = \left|\left\langle\left(\hat{A}-\langle A\rangle\right)\Psi\right|\left(\hat{B}-\langle B\rangle\right)\Psi\right\rangle\right|^2 \tag{0.10}$$

$$= \left|\left\langle\Psi\left|\left(\hat{A}-\langle A\rangle\right)\left(\hat{B}-\langle B\rangle\right)\right|\Psi\right\rangle\right|^2 \tag{0.11}$$

Defining the operators

$$\hat{\Omega} \equiv \hat{A}-\langle A\rangle \tag{0.12}$$

$$\hat{\Lambda} \equiv \hat{B}-\langle B\rangle \tag{0.13}$$

we have

$$|\langle f|g\rangle|^2 = \left|\left\langle\Psi\left|\hat{\Omega}\hat{\Lambda}\right|\Psi\right\rangle\right|^2 \tag{0.14}$$

$$= \frac{1}{4}\left|\left\langle\Psi\left|\left[\hat{\Omega},\hat{\Lambda}\right]_+ + \left[\hat{\Omega},\hat{\Lambda}\right]\right|\Psi\right\rangle\right|^2 \tag{0.15}$$

where

$$\left[\hat{\Omega},\hat{\Lambda}\right]_+ \equiv \hat{\Omega}\hat{\Lambda}+\hat{\Lambda}\hat{\Omega} \tag{0.16}$$

is the anticommutator. For two Hermitian operators, the commutator is the difference between a value and its complex conjugate, so is always pure imaginary (and thus the anticommutator is always real), so we can write this as

$$\left[\hat{\Omega},\hat{\Lambda}\right] = i\Gamma \tag{0.17}$$

for some Hermitian operator $\Gamma$. Using the triangle inequality, we thus arrive at

$$\sigma_A^2\sigma_B^2 \ge |\langle f|g\rangle|^2 \ge \frac{1}{4}\left\langle\Psi\left|\left[\hat{\Omega},\hat{\Lambda}\right]_+\right|\Psi\right\rangle^2 + \frac{1}{4}\langle\Psi|\Gamma|\Psi\rangle^2 \tag{0.18}$$

Comparing this with Griffiths's result, he had

$$\sigma_A^2\sigma_B^2 \ge \left(\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right)^2 = \frac{1}{4}\langle\Psi|\Gamma|\Psi\rangle^2 \tag{0.19}$$

That is, Griffiths's uncertainty principle is actually weaker than Shankar's as he includes only the last term in 0.18. For canonically conjugate operators (such as $X$ and $P$) the commutator is always

$$[X,P] = i\hbar \tag{0.20}$$

so the last term in 0.18 is always $\hbar^2/4$ for any wave function $\Psi$. The first term in 0.18, which involves the anticommutator, will, in general, depend on the wave function $\Psi$, but it is always positive (or zero), so we can still state that, for such operators

$$\sigma_A^2 \sigma_B^2 \geq \frac{\hbar^2}{4} \tag{0.21}$$