DIRECT PRODUCT OF TWO VECTOR SPACES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercise 10.1.1.

Although we've studied quantum systems of more than one particle before (for example, systems of fermions and bosons) as covered by Griffiths's book, the wave functions associated with such particles were just given as products of single-particle wave functions (or linear combinations of these products). We didn't examine the linear algebra behind these functions. In his chapter 10, Shankar begins by describing the algebra of a direct product vector space, so we'll review this here.

The physics begins with an extension of the postulate of quantum mechanics that, for a single particle, the position and momentum obey the commutation relation

$$[X,P]=i\hbar \tag{0.1}$$

To extend this to multi-particle systems, we propose

$$\left[X_i,P_j\right] = i\hbar\delta_{ij} \tag{0.2}$$

$$\left[X_i,X_j\right] = \left[P_i,P_j\right]=0 \tag{0.3}$$

where the subscripts refer to the particle we're considering.

These postulates are translations of the classical Poisson brackets from classical mechanics, following the prescription that to obtain the quantum commutator, we multiply the classical Poisson bracket by $i\hbar$. The physics in these relations is that properties such as position or momentum of different particles are simultaneously observable, although the position and momentum of a single particle are still governed by the uncertainty principle.

We'll now restrict our attention to a two-particle system. In such a system, the eigenstate of the position operators is written as $\left|x_1x_2\right\rangle$ and satisfies the eigenvalue equation

$$X_i\left|x_1x_2\right\rangle = x_i\left|x_1x_2\right\rangle \tag{0.4}$$

Operators referring to particle i effectively ignore any quantities associated with the other particle.

So what exactly are these states $|x_1x_2\rangle$? They are a set of vectors that span a Hilbert space that describes the state of two particles. Note that we can use any two commuting operators $\Omega_1(X_1,P_1)$ and $\Omega_2(X_2,P_2)$ to create a set of eigenkets $|\omega_1\omega_2\rangle$ which also span the space. Any operator that is a function of the position and momentum of only one of the particles always commutes with a similar operator that is a function of only the other particle, since the position and momentum operators of which it is a function commute with those of the other operator. That is

$$[\Omega(X_1,P_1),\Lambda(X_2,P_2)]=0 \tag{0.5}$$

The space spanned by $|x_1x_2\rangle$ can also be written as a *direct product* of two one-particle spaces. This space is written as $\mathbb{V}_{1\otimes2}$ where the symbol $\otimes$ is the direct product symbol (it's also the logo of the X-Men, but we won't pursue that). The direct product is composed of the two single-particle spaces $\mathbb{V}_1$ (spanned by $|x_1\rangle$) and $\mathbb{V}_2$ (spanned by $|x_2\rangle$). The notation gets quite cumbersome at this point, so let's spell it out carefully. For an operator Ω, we can specify which *particle* it acts on by a subscript, and which *space* it acts on by a superscript. Thus $X_1^{(1)}$ is the position operator for particle 1, which operates on the vector space $\mathbb{V}_1$. It might seem redundant at this point to specify both the particle and the space, since it would seem that these are always the same. However, be patient...

From the two one-particle spaces, we can form the two-particle space by taking the direct product of the two one-particle states. Thus the state in which particle 1 is in state $|x_1\rangle$ and particle 2 is in state $|x_2\rangle$ is written as

$$|x_1x_2\rangle=|x_1\rangle\otimes|x_2\rangle \tag{0.6}$$

It is important to note that this object is composed of two vectors from *different* vector spaces. The inner and outer products we've dealt with up to now, for things like finding the probability that a state has a particular value and so on, that is, objects like $\langle\psi_1|\psi_2\rangle$ and $|\psi_1\rangle\langle\psi_2|$, are composed of two vectors from the *same* vector space, so no direct product is needed.

If we recall the direct sum of two vector spaces

$$\mathbb{V}_{1\oplus2}=\mathbb{V}_1\oplus\mathbb{V}_2 \tag{0.7}$$

in that case, the dimension of $\mathbb{V}_{1\oplus2}$ is the sum of the dimensions of $\mathbb{V}_1$ and $\mathbb{V}_2$. For a direct product we see from 0.6 that for each vector $|x_1\rangle$ there is one basis vector for each vector $|x_2\rangle$. Thus the number of basis vectors is the *product* of the number of basis vectors in each of the two one-particle spaces. In other words, the dimension of a direct product is the product of the dimensions of the two vector spaces of which it is composed. [In the case here, both the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ have infinite dimension, so the dimension of $\mathbb{V}_{1\otimes 2}$ is in effect, 'doubly infinite'. In a case where $\mathbb{V}_1$ and $\mathbb{V}_2$ have finite dimension, we can then just multiply these dimensions to get the dimension of $\mathbb{V}_{1\otimes 2}$.]

As $\mathbb{V}_{1\otimes 2}$ is a vector space with basis vectors $|x_1\rangle \otimes |x_2\rangle$, any linear combination of the basis vectors is also a vector in the space $\mathbb{V}_{1\otimes 2}$. Thus the vector

$$|\psi\rangle = |x_1\rangle \otimes |x_2\rangle + |y_1\rangle \otimes |y_2\rangle \tag{0.8}$$

is in $\mathbb{V}_{1\otimes 2}$, although it can't be written as a direct product of the two one-particle spaces $\mathbb{V}_1$ and $\mathbb{V}_2$.

Having defined the direct product space, we now need to consider operators in this space. Although Shankar states that it 'is intuitively clear' that a single particle operator such as $X_1^{(1)}$ must have a corresponding operator in the product space that has the same effect has $X_1^{(1)}$ has on the single particle state, it seems to me to be more of a postulate. In any case, it is proposed that if

$$X_1^{(1)} |x_1\rangle = x_1 |x_1\rangle \tag{0.9}$$

then in the product space there must be an operator $X_1^{(1)\otimes(2)}$ that operates only on particle 1, with the same effect, that is

$$X_1^{(1)\otimes(2)} |x_1\rangle \otimes |x_2\rangle = x_1 |x_1\rangle \otimes |x_2\rangle \tag{0.10}$$

The notation can be explained as follows. The subscript 1 in $X_1^{(1)\otimes(2)}$ means that the operator operates on particle 1, while the superscript $(1)\otimes(2)$ means that the operator operates in the product space $\mathbb{V}_{1\otimes 2}$. In effect, the operator $X_1^{(1)\otimes(2)}$ is the product of two one-particle operators $X_1^{(1)}$, which operates on space $\mathbb{V}_1$ and an identity operator $I_2^{(2)}$ which operates on space $\mathbb{V}_2$. That is, we can write

$$X_1^{(1)\otimes(2)} = X_1^{(1)} \otimes I_2^{(2)} \tag{0.11}$$

$$X_1^{(1)\otimes(2)} |x_1\rangle \otimes |x_2\rangle = \left|X_1^{(1)} x_1\right\rangle \otimes \left|I_2^{(2)} x_2\right\rangle \tag{0.12}$$

$$= x_1 |x_1\rangle \otimes |x_2\rangle \tag{0.13}$$

Generally, if we have two one-particle operators $\Gamma_1^{(1)}$ and $\Lambda_2^{(2)}$, each of which operates on a different one-particle state, then we can form a direct product operator with the property

$$\left(\Gamma_1^{(1)} \otimes \Lambda_2^{(2)}\right) |\omega_1\rangle \otimes |\omega_2\rangle = \left|\Gamma_1^{(1)} \omega_1\right\rangle \otimes \left|\Lambda_2^{(2)} \omega_2\right\rangle \tag{0.14}$$

That is, a single-particle operator that operates on space i that forms part of a direct product operator operates only on the factor of a direct product vector that corresponds to the one-particle space. Given this property, it's fairly easy to derive a few properties of direct product operators.

(0.15)
$$\left[\Omega_1^{(1)} \otimes I^{(2)}, I^{(1)} \otimes \Lambda_2^{(2)}\right] |\omega_1\rangle \otimes |\omega_2\rangle = \Omega_1^{(1)} \otimes I^{(2)} I^{(1)} \otimes \Lambda_2^{(2)} |\omega_1\rangle \otimes |\omega_2\rangle -$$
$$I^{(1)} \otimes \Lambda_2^{(2)} \Omega_1^{(1)} \otimes I^{(2)} |\omega_1\rangle \otimes |\omega_2\rangle \tag{0.16}$$
$$= \Omega_1^{(1)} \otimes I^{(2)} \left|I^{(1)} \omega_1\right\rangle \otimes \left|\Lambda_2^{(2)} \omega_2\right\rangle - \tag{0.17}$$
$$I^{(1)} \otimes \Lambda_2^{(2)} \left|\Omega_1^{(1)} \omega_1\right\rangle \otimes \left|I^{(2)} \omega_2\right\rangle \tag{0.18}$$
$$= \left|\Omega_1^{(1)} \omega_1\right\rangle \otimes \left|I^{(2)} \Lambda_2^{(2)} \omega_2\right\rangle - \tag{0.19}$$
$$\left|I^{(1)} \Omega_1^{(1)} \omega_1\right\rangle \otimes \left|\Lambda_2^{(2)} \omega_2\right\rangle \tag{0.20}$$
$$= \left|\Omega_1^{(1)} \omega_1\right\rangle \otimes \left|\Lambda_2^{(2)} \omega_2\right\rangle - \left|\Omega_1^{(1)} \omega_1\right\rangle \otimes \left|\Lambda_2^{(2)} \omega_2\right\rangle \tag{0.21}$$
$$= 0 \tag{0.22}$$

This derivation shows that the identity operators effectively cancel out and we're left with the earlier commutator 0.5 between two operators that operate on different spaces.

The next derivation involves the successive operation of two direct product operators.

(0.23)
$$\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle = \left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left|\theta_1^{(1)}\omega_1\right\rangle \otimes \left|\Lambda_2^{(2)}\omega_2\right\rangle$$

(0.24)
$$= \left|\Omega_1^{(1)}\theta_1^{(1)}\omega_1\right\rangle \otimes \left|\Gamma_2^{(2)}\Lambda_2^{(2)}\omega_2\right\rangle$$

(0.25)
$$= \left(\Omega_1^{(1)}\theta_1^{(1)}\right) \otimes \left(\Gamma_2^{(2)}\Lambda_2^{(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle$$

(0.26)
$$= \left\{(\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)}\right\}|\omega_1\rangle \otimes |\omega_2\rangle$$

(0.27)
$$\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) = (\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)}$$

Next, another commutator identity. Given

(0.28)
$$\left[\Omega_1^{(1)}, \Lambda_1^{(1)}\right] = \Gamma_1^{(1)}$$

we have

(0.29)
$$\left[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right]|\omega_1\rangle \otimes |\omega_2\rangle = \left[\Omega_1^{(1)} \otimes I^{(2)}, \Lambda_1^{(1)} \otimes I^{(2)}\right]|\omega_1\rangle \otimes |\omega_2\rangle$$

(0.30)
$$= \left|\left[\Omega_1^{(1)}, \Lambda_1^{(1)}\right]\omega_1\right\rangle \otimes \left|I^{(2)}\omega_2\right\rangle$$

(0.31)
$$= \left|\Gamma_1^{(1)}\omega_1\right\rangle \otimes \left|I^{(2)}\omega_2\right\rangle$$

(0.32)
$$= \Gamma_1^{(1)} \otimes I^{(2)}|\omega_1\rangle \otimes |\omega_2\rangle$$

(0.33)
$$\left[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right] = \Gamma_1^{(1)} \otimes I^{(2)}$$

Finally, the square of the sum of two operators:

(0.34)
$$\left(\Omega_1^{(1)\otimes(2)}+\Omega_2^{(1)\otimes(2)}\right)^2|\omega_1\rangle\otimes|\omega_2\rangle=\left(\Omega_1^{(1)}\otimes I^{(2)}+I^{(1)}\otimes\Omega_2^{(2)}\right)^2|\omega_1\rangle\otimes|\omega_2\rangle$$

$$=\left(\Omega_1^{(1)}\otimes I^{(2)}\right)^2|\omega_1\rangle\otimes|\omega_2\rangle+ \tag{0.35}$$

$$\Omega_1^{(1)}\otimes I^{(2)}I^{(1)}\otimes\Omega_2^{(2)}|\omega_1\rangle\otimes|\omega_2\rangle+ \tag{0.36}$$

$$I^{(1)}\otimes\Omega_2^{(2)}\Omega_1^{(1)}\otimes I^{(2)}|\omega_1\rangle\otimes|\omega_2\rangle+ \tag{0.37}$$

$$\left(I^{(1)}\otimes\Omega_2^{(2)}\right)^2|\omega_1\rangle\otimes|\omega_2\rangle \tag{0.38}$$

$$=\left|\left(\Omega_1^2\right)^{(1)}\omega_1\right\rangle\otimes\left|I^{(2)}\omega_2\right\rangle+ \tag{0.39}$$

$$\left|\Omega_1^{(1)}\omega_1\right\rangle\otimes\left|\Omega_2^{(2)}\omega_2\right\rangle+ \tag{0.40}$$

$$\left|\Omega_1^{(1)}\omega_1\right\rangle\otimes\left|\Omega_2^{(2)}\omega_2\right\rangle+ \tag{0.41}$$

$$\left|I^{(1)}\omega_1\right\rangle\otimes\left|\left(\Omega_2^2\right)^{(2)}\omega_2\right\rangle \tag{0.42}$$

$$=\left(\left(\Omega_1^2\right)^{(1)}\otimes I^{(2)}+2\Omega_1^{(1)}\otimes\Omega_2^{(2)}+I^{(1)}\otimes\left(\Omega_2^2\right)^{(2)}\right)|\omega_1\rangle\otimes|\omega_2\rangle \tag{0.43}$$

(0.44)
$$\left(\Omega_1^{(1)\otimes(2)}+\Omega_2^{(1)\otimes(2)}\right)^2=\left(\Omega_1^2\right)^{(1)}\otimes I^{(2)}+2\Omega_1^{(1)}\otimes\Omega_2^{(2)}+I^{(1)}\otimes\left(\Omega_2^2\right)^{(2)}$$

In this derivation, we used the fact that the identity operator leaves its operand unchanged, and thus that $\left(I^2\right)^{(i)}=I^{(i)}$ for either space i.

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