DIRECT PRODUCT OF VECTOR SPACES: 2-DIM EXAMPLES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercise 10.1.2.

To help with understanding the direct product of two vector spaces, some examples with a couple of 2-d vector spaces are useful. Suppose the one-particle Hilbert space is two-dimensional, with basis vectors $|+\rangle$ and $|-\rangle$. Now suppose we have two such particles, each in its own 2-d space, $\mathbb{V}_1$ for particle 1 and $\mathbb{V}_2$ for particle 2. We can define a couple of operators by their matrix elements in these two spaces. We define

$$\sigma_1^{(1)} \equiv \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \tag{1}$$

$$\sigma_2^{(2)} \equiv \left[\begin{array}{cc} e & f \\ g & h \end{array}\right] \tag{2}$$

where the first column and row refer to basis vector $|+\rangle$ and the second column and row to $|-\rangle$. Recall that the subscript on each σ refers to the particle and the superscript refers to the vector space. Thus $\sigma_1^{(1)}$ is an operator in space $\mathbb{V}_1$ for particle 1.

Now consider the direct product space $\mathbb{V}_1 \otimes \mathbb{V}_2$, which is spanned by the four basis vectors formed by direct products of the two basis vectors in each of the one-particle spaces, that is by $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$ and $|-\rangle \otimes |-\rangle$. Each of the σ operators has a corresponding version in the product space, which is formed by taking the direct product of the one-particle version for one of the particles with the identity operator for the other particle. That is

$$\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes I^{(2)} \tag{3}$$

$$\sigma_2^{(1)\otimes(2)} = I^{(1)} \otimes \sigma_2^{(2)} \tag{4}$$

To get the matrix elements in the product space, we need the form of the identity operators in the one-particle spaces. They are, as usual

$$I^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

$$I^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

I've written the two identity operators as separate equations since although they have the same numerical form as a matrix, the two operators operate on different spaces, so they are technically different operators. To get the matrix elements of $\sigma_1^{(1)\otimes(2)}$ we can expand the direct product (Shankar suggests using the 'method of images', although I have no idea what this is. I doubt that it's the same method of images used in electrostatics, and Google draws a blank for any other kind of method of images.) In any case, we can form the product by taking the corresponding matrix elements. For example

$$\left\langle ++\left|\sigma_1^{(1)\otimes(2)}\right|++\right\rangle = \left(\langle+|\otimes\langle+|\right)\sigma_1^{(1)}\otimes I^{(2)}\left(|+\rangle\otimes|+\rangle\right) \tag{7}$$

$$= \left\langle +\left|\sigma_1^{(1)}\right|+\right\rangle\left\langle +\left|I^{(2)}\right|+\right\rangle \tag{8}$$

$$= a\times 1 = a \tag{9}$$

When working out the RHS of the first line, remember that operators with a superscript (1) operate only on bras and kets from the space $\mathbb{V}_1$ and operators with a superscript (2) operate only on bras and kets from the space $\mathbb{V}_2$. Applying the same technique for the remaining elements gives

$$\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)}\otimes I^{(2)} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \tag{10}$$

Another less tedious way of getting this result is to note that we can form the direct product by taking each element in the first matrix $\sigma_1^{(1)}$ from 1 and multiply it into the second matrix $I^{(2)}$ from 6. Thus the top 2×2 elements in $\sigma_1^{(1)\otimes(2)}$ are obtained by taking the element $\left\langle +\left|\sigma_1^{(1)}\right|+\right\rangle = a$ from 1 and multiplying it into the matrix $I^{(2)}$ from 6. That is, the upper left 2×2 block is formed from

$$aI_{2\times 2}^{(2)} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{11}$$

and so on for the other three 2×2 blocks in the complete matrix. Note that it's important to get things in the right order, as the direct product is not commutative.

To get the other direct product, we can apply the same technique:

$$\sigma_2^{(1)\otimes(2)} = I^{(1)}\otimes\sigma_2^{(2)} = \begin{bmatrix} e & f & 0 & 0\\ g & h & 0 & 0\\ 0 & 0 & e & f\\ 0 & 0 & g & h \end{bmatrix} \tag{12}$$

Again, note that

$$I^{(1)}\otimes\sigma_2^{(2)} \neq \sigma_2^{(2)}\otimes I^{(1)} = \begin{bmatrix} e & 0 & f & 0\\ 0 & e & 0 & f\\ g & 0 & h & 0\\ 0 & g & 0 & h \end{bmatrix} \tag{13}$$

Finally, we can work out the direct product version of the product of two one-particle operators. That is, we want

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)}\otimes\sigma_2^{(2)} \tag{14}$$

We can do this in two ways. First, we can apply the same recipe as in the previous example. We take each element of $\sigma_1^{(1)}$ and multiply it into the full matrix $\sigma_2^{(2)}$:

$$\sigma_1^{(1)}\otimes\sigma_2^{(2)} = \begin{bmatrix} ae & af & be & bf\\ ag & ah & bg & bh\\ ce & cf & de & df\\ cg & ch & dg & dh \end{bmatrix} \tag{15}$$

Second, we can take the matrix product of $\sigma_1^{(1)\otimes(2)}$ from 10 with $\sigma_2^{(1)\otimes(2)}$ from 12:

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \begin{bmatrix} a & 0 & b & 0\\ 0 & a & 0 & b\\ c & 0 & d & 0\\ 0 & c & 0 & d \end{bmatrix}\begin{bmatrix} e & f & 0 & 0\\ g & h & 0 & 0\\ 0 & 0 & e & f\\ 0 & 0 & g & h \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} ae & af & be & bf\\ ag & ah & bg & bh\\ ce & cf & de & df\\ cg & ch & dg & dh \end{bmatrix} \tag{17}$$