

HARMONIC OSCILLATOR IN 2-D AND 3-D, AND IN POLAR AND SPHERICAL COORDINATES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercises 10.2.2 - 10.2.3.

We've seen that the 3-d isotropic harmonic oscillator can be solved in rectangular coordinates using separation of variables. The Hamiltonian is

$$(0.1) \quad H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2 + z^2)$$

The solution to the Schrödinger equation is just the product of three one-dimensional oscillator eigenfunctions, one for each coordinate. That is

$$(0.2) \quad \psi_n(x, y, z) = \psi_{n_x}(x) \psi_{n_y}(y) \psi_{n_z}(z)$$

Each one-dimensional eigenfunction can be expressed in terms of Hermite polynomials as

$$(0.3) \quad \psi_{n_x}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n_x} n_x!}} H_{n_x} \left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-m\omega x^2/2\hbar}$$

with the functions for y and z obtained by replacing x by y or z and n_x by n_y or n_z . We also saw earlier that in the 3-d oscillator, the total energy for state $\psi_n(x, y, z)$ is given in terms of the quantum numbers of the three 1-d oscillators as

$$(0.4) \quad E_n = \hbar\omega \left(n + \frac{3}{2}\right) = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2}\right)$$

and that the degeneracy of level n is $\frac{1}{2}(n+1)(n+2)$.

Since the Hermite polynomial H_{n_x} has parity $(-1)^{n_x}$ (that is, odd (even) polynomials are odd (even) functions), the 3-d wave function ψ_n has parity $(-1)^{n_x} (-1)^{n_y} (-1)^{n_z} = (-1)^n$.

We can write the one $n = 0$ state and three $n = 1$ states in spherical coordinates using the standard transformation

$$(0.5) \quad x = r \sin \theta \cos \phi$$

$$(0.6) \quad y = r \sin \theta \sin \phi$$

$$(0.7) \quad z = r \cos \theta$$

Using the notation $\psi_n = \psi_{n_x n_y n_z} = \psi_{n_x} \psi_{n_y} \psi_{n_z}$, we have, using $H_0(y) = 1$ and $H_1(y) = 2y$:

$$(0.8) \quad \psi_{000} = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar}$$

$$(0.9) \quad \psi_{100} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin \theta \cos \phi$$

$$(0.10) \quad \psi_{010} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin \theta \sin \phi$$

$$(0.11) \quad \psi_{001} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \cos \theta$$

We can check that these are the correct spherical versions of the eigenfunctions by using the Schrödinger equation in spherical coordinates, which is

$$(0.12) \quad H\psi = \left[-\frac{\hbar^2 \nabla^2}{2m} + \frac{m\omega^2}{2} r^2 \right] \psi = E\psi$$

The spherical laplacian operator is

$$(0.13) \quad \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

You can grind through the derivatives by hand if you like, but I just used Maple to do it, giving the results

$$(0.14) \quad H\psi_{000} = \frac{3}{2} \hbar \omega \psi_{000}$$

$$(0.15) \quad H\psi_{100} = \frac{5}{2} \hbar \omega \psi_{100}$$

$$(0.16) \quad H\psi_{010} = \frac{5}{2} \hbar \omega \psi_{010}$$

$$(0.17) \quad H\psi_{001} = \frac{5}{2} \hbar \omega \psi_{001}$$

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In two dimensions, the analysis is pretty much the same. In the more general case where the masses are equal, but $\omega_x \neq \omega_y$, the Hamiltonian is

$$(0.18) \quad H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2)$$

A solution by separation of variables still works, with the result

$$(0.19) \quad \Psi_n(x, y) = \Psi_{n_x}(x) \Psi_{n_y}(y)$$

The total energy is

$$(0.20) \quad E_n = E_{n_x} + E_{n_y} = \hbar\omega \left(n_x + \frac{1}{2} + n_y + \frac{1}{2} \right) = \hbar\omega (n + 1)$$

For a given energy level $n = n_x + n_y$, there are $n + 1$ ways of forming n out of a sum of 2 non-negative integers, so the degeneracy of level n is $n + 1$.

The one $n = 0$ state and two $n = 1$ states are

$$(0.21) \quad \Psi_{00} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar}$$

$$(0.22) \quad \Psi_{10} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar} x$$

$$(0.23) \quad \Psi_{01} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar} y$$

To translate to polar coordinates, we use the transformations

$$(0.24) \quad x = \rho \cos \phi$$

$$(0.25) \quad y = \rho \sin \phi$$

so we have

$$(0.26) \quad \Psi_{00} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar}$$

$$(0.27) \quad \Psi_{10} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar} \rho \cos \phi$$

$$(0.28) \quad \Psi_{01} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar} \rho \sin \phi$$

Again, we can check this by plugging these polar formulas into the polar Schrödinger equation, where the 2-d Laplacian is

$$(0.29) \quad \nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}$$

The results are

$$(0.30) \quad H\psi_{00} = \hbar\omega\psi_{00}$$

$$(0.31) \quad H\psi_{10} = 2\hbar\omega\psi_{10}$$

$$(0.32) \quad H\psi_{01} = 2\hbar\omega\psi_{01}$$

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