HARMONIC OSCILLATOR IN 2-D AND 3-D, AND IN POLAR AND SPHERICAL COORDINATES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercises 10.2.2 - 10.2.3.

We've seen that the 3-d isotropic harmonic oscillator can be solved in rectangular coordinates using separation of variables. The Hamiltonian is

$$H=\frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{m\omega^2}{2}\left(x^2+y^2+z^2\right) \tag{1}$$

The solution to the Schrödinger equation is just the product of three one-dimensional oscillator eigenfunctions, one for each coordinate. That is

$$\psi_n(x,y,z)=\psi_{n_x}(x)\,\psi_{n_y}(y)\,\psi_{n_z}(z) \tag{2}$$

Each one-dimensional eigenfunction can be expressed in terms of Hermite polynomials as

$$\psi_{n_x}(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{n_x}n_x!}}H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar} \tag{3}$$

with the functions for y and z obtained by replacing x by y or z and n_x by n_y or n_z. We also saw earlier that in the 3-d oscillator, the total energy for state $\psi_n(x,y,z)$ is given in terms of the quantum numbers of the three 1-d oscillators as

$$E_n=\hbar\omega\left(n+\frac{3}{2}\right)=\hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right) \tag{4}$$

and that the degeneracy of level n is $\frac{1}{2}(n+1)(n+2)$.

Since the Hermite polynomial H_{n_x} has parity $(-1)^{n_x}$ (that is, odd (even) polynomials are odd (even) functions), the 3-d wave function ψ_n has parity $(-1)^{n_x}(-1)^{n_y}(-1)^{n_z}=(-1)^n$.

We can write the one $n=0$ state and three $n=1$ states in spherical coordinates using the standard transformation

$$\begin{aligned} x &= r\sin\theta\cos\phi && (5)\\ y &= r\sin\theta\sin\phi && (6)\\ z &= r\cos\phi && (7) \end{aligned}$$

Using the notation $\psi_n = \psi_{n_x n_y n_z} = \psi_{n_x}\psi_{n_y}\psi_{n_z}$, we have, using $H_0(y) = 1$ and $H_1(y) = 2y$:

$$\psi_{000} = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} \tag{8}$$

$$\psi_{100} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r\sin\theta\cos\phi \tag{9}$$

$$\psi_{010} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r\sin\theta\sin\phi \tag{10}$$

$$\psi_{001} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r\cos\theta \tag{11}$$

We can check that these are the correct spherical versions of the eigenfunctions by using the Schrödinger equation in spherical coordinates, which is

$$H\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + \frac{m\omega^2}{2}r^2\right]\psi = E\psi \tag{12}$$

The spherical laplacian operator is

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{13}$$

You can grind through the derivatives by hand if you like, but I just used Maple to do it, giving the results

$$\begin{aligned} H\psi_{000} &= \frac{3}{2}\hbar\omega\psi_{000} && (14)\\ H\psi_{100} &= \frac{5}{2}\hbar\omega\psi_{100} && (15)\\ H\psi_{010} &= \frac{5}{2}\hbar\omega\psi_{010} && (16)\\ H\psi_{001} &= \frac{5}{2}\hbar\omega\psi_{001} && (17) \end{aligned}$$

In two dimensions, the analysis is pretty much the same. In the more general case where the masses are equal, but $\omega_x \neq \omega_y$, the Hamiltonian is

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) \tag{18}$$

A solution by separation of variables still works, with the result

$$\psi_n(x,y) = \psi_{n_x}(x)\,\psi_{n_y}(y) \tag{19}$$

The total energy is

$$E_n = E_{n_x} + E_{n_y} = \hbar\omega\left(n_x + \frac{1}{2} + n_y + \frac{1}{2}\right) = \hbar\omega(n+1) \tag{20}$$

For a given energy level $n = n_x + n_y$, there are $n+1$ ways of forming n out of a sum of 2 non-negative integers, so the degeneracy of level n is $n+1$.

The one $n=0$ state and two $n=1$ states are

$$\psi_{00} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\left(x^2+y^2\right)/2\hbar} \tag{21}$$

$$\psi_{10} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\left(x^2+y^2\right)/2\hbar} x \tag{22}$$

$$\psi_{01} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\left(x^2+y^2\right)/2\hbar} y \tag{23}$$

To translate to polar coordinates, we use the transformations

$$x = \rho\cos\phi \tag{24}$$

$$y = \rho\sin\phi \tag{25}$$

so we have

$$\psi_{00} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\rho^2/2\hbar} \tag{26}$$

$$\psi_{10} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\rho^2/2\hbar}\rho\cos\phi \tag{27}$$

$$\psi_{01} = \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega\rho^2/2\hbar}\rho\sin\phi \tag{28}$$

Again, we can check this by plugging these polar formulas into the polar Schrödinger equation, where the 2-d Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} \tag{29}$$

The results are

$$
\begin{aligned}
H\psi_{00} &= \hbar\omega\psi_{00} && (30)\\
H\psi_{10} &= 2\hbar\omega\psi_{10} && (31)\\
H\psi_{01} &= 2\hbar\omega\psi_{01} && (32)
\end{aligned}
$$

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