INVARIANCE OF SYMMETRIC AND ANTISYMMETRIC STATES; EXCHANGE OPERATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercise 10.3.5.

In a system with two particles, the state in the X basis is given by $|x_1,x_2\rangle$ where x_i is the position of particle i. We can define the exchange operator P_{12} as an operator that swaps the two particles, so that

$$P_{12}|x_1,x_2\rangle = |x_2,x_1\rangle \tag{1}$$

To find the eigenvalues and eigenvectors of P_{12} we have

$$P_{12}|\psi(x_1,x_2)\rangle = \alpha|\psi(x_1,x_2)\rangle = \psi(x_2,x_1) \tag{2}$$

where α is the eigenvalue and $|\psi(x_1,x_2)\rangle$ is the eigenvector. Using the same argument as before, we can write

$$|\psi(x_1,x_2)\rangle = \beta|x_1,x_2\rangle + \gamma|x_2,x_1\rangle \tag{3}$$

$$|\psi(x_2,x_1)\rangle = \beta|x_2,x_1\rangle + \gamma|x_1,x_2\rangle \tag{4}$$

$$= \alpha[\beta|x_1,x_2\rangle + \gamma|x_2,x_1\rangle] \tag{5}$$

Equating coefficients in the first and third lines, we arrive at

$$\alpha = \pm 1 \tag{6}$$

which gives the same symmetric and antisymmetric eigenfunctions that we had before:

$$\psi_S(x_1,x_2) = \frac{1}{\sqrt{2}}(|x_1,x_2\rangle + |x_2,x_1\rangle) \tag{7}$$

$$\psi_A(x_1,x_2) = \frac{1}{\sqrt{2}}(|x_1,x_2\rangle - |x_2,x_1\rangle) \tag{8}$$

We can derive a couple of other properties of the exchange operator by noting that if it is applied twice in succession, we get the original state back, so that

$$P_{12}^2 = I \tag{9}$$

$$P_{12} = P_{12}^{-1} \tag{10}$$

Thus the operator is its own inverse.

Consider also the two states $|x_1',x_2'\rangle$ and $|x_1,x_2\rangle$. Then

$$\left\langle x_1',x_2'\left|P_{12}^{\dagger}P_{12}\right|x_1,x_2\right\rangle = \left\langle P_{12}x_1',x_2'\left|P_{12}x_1,x_2\right.\right\rangle \tag{11}$$

$$= \left\langle x_2',x_1'\left|x_2,x_1\right.\right\rangle \tag{12}$$

$$= \left(\left\langle x_2'\right|\otimes\left\langle x_1'\right|\right)\left(|x_2\rangle\otimes|x_1\rangle\right) \tag{13}$$

$$= \delta\left(x_2'-x_2\right)\delta\left(x_1'-x_1\right) \tag{14}$$

However, the last line is just equal to the inner product of the original states, that is

$$\left\langle x_1',x_2'\left|x_1,x_2\right.\right\rangle = \delta\left(x_2-x_2'\right)\delta\left(x_1-x_1'\right) = \delta\left(x_2'-x_2\right)\delta\left(x_1'-x_1\right) \tag{15}$$

This means that

$$P_{12}^{\dagger}P_{12} = I \tag{16}$$

$$P_{12}^{\dagger} = P_{12}^{-1} = P_{12} \tag{17}$$

Thus P_{12} is both Hermitian and unitary.

Shankar asks us to show that, for a general basis vector $|\omega_1,\omega_2\rangle$, $P_{12}|\omega_1,\omega_2\rangle = |\omega_2,\omega_1\rangle$. One argument could be that, since the X basis spans the space, we can express any other vector such as $|\omega_1,\omega_2\rangle$ as a linear combination of the $|x_1,x_2\rangle$ vectors, so that applying P_{12} to $|\omega_1,\omega_2\rangle$ means applying it to a sum of $|x_1,x_2\rangle$ vectors, which swaps the two particles in every term. I'm not sure if this is a rigorous result. In any case, if we accept this result it shows that if we start in a state that is totally symmetric (that is, a boson state), this state is an eigenvector of P_{12} with eigenvalue $+1$. Similarly, if we start in an antisymmetric (fermion) state, this state is an eigenvector of P_{12} with eigenvalue -1.

Now we can look at some other properties of P_{12}. Consider

$$P_{12}X_1P_{12}|x_1,x_2\rangle = P_{12}X_1|x_2,x_1\rangle \tag{18}$$

$$= x_2P_{12}|x_2,x_1\rangle \tag{19}$$

$$= x_2|x_1,x_2\rangle \tag{20}$$

$$= X_2|x_1,x_2\rangle \tag{21}$$

This follows because the operator X_1 operates on the first particle in the state $|x_2,x_1\rangle$ which on the RHS of the first line is at position x_2. Thus $X_1|x_2,x_1\rangle = x_2|x_2,x_1\rangle$, that is, X_1 returns the numerical value of the position of the first particle, which is x_2. This means that in terms of the operators alone

$$P_{12}X_1P_{12} = X_2 \quad (22)$$

$$P_{12}X_2P_{12} = X_1 \quad (23)$$

$$P_{12}P_1P_{12} = P_2 \quad (24)$$

$$P_{12}P_2P_{12} = P_1 \quad (25)$$

In the last two lines, the operator P_i is the momentum of particle i, and the result follows by applying the operators to the momentum basis state $|p_1,p_2\rangle$.

For some general operator which can be expanded in a power series of terms containing powers of X_i and/or P_i, we can use 10 to insert $P_{12}P_{12}$ between every factor of X_i or P_i. For example

$$P_{12}P_1X_2^2X_1P_{12} = P_{12}P_1P_{12}P_{12}X_2P_{12}P_{12}X_2P_{12}P_{12}X_1P_{12} \quad (26)$$

$$= P_2X_1^2X_2 \quad (27)$$

That is, for any operator $\Omega(X_1,P_1;X_2,P_2)$ we have

$$P_{12}\Omega(X_1,P_1;X_2,P_2)P_{12} = \Omega(X_2,P_2;X_1,P_1) \quad (28)$$

The Hamiltonian for a system of two *identical* particles must be symmetric under exchange of the particles, since it represents an observable (the energy), and this observable must remain unchanged if we swap the particles. (In the case of two fermions, the wave function is antisymmetric, but the wave function itself is not an observable. The wave function gets multiplied by -1 if we swap the particles, but the square modulus of the wave function, which contains the physics, remains the same.) Thus we have

$$P_{12}H(X_1,P_1;X_2,P_2)P_{12} = H(X_2,P_2;X_1,P_1) = H(X_1,P_1;X_2,P_2) \quad (29)$$

[Note that this condition doesn't necessarily follow if the two particles are not identical, since exchanging them in this case leads to an observably different system. For example, exchanging the proton and electron in a hydrogen atom leads to a different system.]

The propagator is defined as

$$U(t) = e^{-iHt/\hbar} \tag{30}$$

and the propagator dictates how a state evolves according to

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \tag{31}$$

Since the only operator on which U depends is H, then U is also invariant, so that

$$P_{12}U(X_1,P_1;X_2,P_2)P_{12} = U(X_2,P_2;X_1,P_1) = U(X_1,P_1;X_2,P_2) \tag{32}$$

Multiplying from the left by P_{12} and subtracting, we get the commutator

$$[U,P_{12}] = 0 \tag{33}$$

For a symmetric state $|\psi_S\rangle$ or antisymmetric state $|\psi_A\rangle$, we have

$$UP_{12}|\psi_S(0)\rangle = U|\psi_S(0)\rangle = |\psi_S(t)\rangle = P_{12}U|\psi_S(0)\rangle \tag{34}$$

$$UP_{12}|\psi_A(0)\rangle = -U|\psi_A(0)\rangle = -|\psi_A(t)\rangle = P_{12}U|\psi_A(0)\rangle \tag{35}$$

This means that states that begin as symmetric or antisymmetric remain symmetric or antisymmetric for all time. In other words, a system that starts in an eigenstate of P_{12} remains in the same eigenstate as time passes.