

## CORRESPONDENCE BETWEEN CLASSICAL AND QUANTUM TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11.

When we consider infinitesimal transformations of some dynamical variable, there is a correspondence between classical and quantum mechanics which we can see as follows. First, we'll summarize the results from classical mechanics. We can define a canonical transformation generated by a variable  $g$  as

$$\bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i \quad (1)$$

$$\bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i \quad (2)$$

Here,  $\varepsilon$  is an infinitesimal amount and  $\delta q_i$  and  $\delta p_i$  are the infinitesimal amounts by which the coordinates and momenta vary. It follows from these definitions that, for any dynamical variable  $\omega$ , its variation  $\delta\omega$  is given by a Poisson bracket

$$\delta\omega = \omega(\bar{q}_i, \bar{p}_i) - \omega(q_i, p_i) = \varepsilon \{\omega, g\} \quad (3)$$

For the special cases of coordinates and momenta, this is

$$\delta q_i = \varepsilon \{q_i, g\} \quad (4)$$

$$\delta p_i = \varepsilon \{p_i, g\} \quad (5)$$

If the generator is the momentum  $p_j$ , then

$$\delta q_i = \varepsilon \{q_i, p_j\} = \varepsilon \delta_{ij} \quad (6)$$

$$\delta p_i = \varepsilon \{p_i, p_j\} = 0 \quad (7)$$

Thus, in classical mechanics,  $p_j$  is the generator of translations in direction  $j$ .

If  $\omega = H$  (the Hamiltonian) and if  $\{H, g\} = 0$ , then  $g$  is conserved (it doesn't vary with time). Because the transformation 1 and 2 is canonical, it preserves the Poisson brackets so that

$$\{\bar{q}_i, \bar{q}_j\} = \{\bar{p}_i, \bar{p}_j\} = 0 \quad (8)$$

$$\{\bar{q}_i, \bar{p}_j\} = \delta_{ij} \quad (9)$$

What do these things correspond to in quantum mechanics? [I find Shankar's treatment in section 11.2 to be almost tautological, since it merely repeats the derivation given earlier. I'll try to be a bit more general.]

Suppose we have some infinitesimal transformation given by a unitary operator  $U(\varepsilon)$ . We can then define the changes in  $X$  and  $P$  by

$$\delta X = U^\dagger(\varepsilon) X U(\varepsilon) - X \quad (10)$$

$$\delta P = U^\dagger(\varepsilon) P U(\varepsilon) - P \quad (11)$$

Since  $U(\varepsilon)$  describes an infinitesimal transformation, we can expand it to first order in  $\varepsilon$ :

$$U(\varepsilon) = I - \frac{i\varepsilon}{\hbar} G \quad (12)$$

where  $G = G^\dagger$  is some Hermitian operator known as the generator of the transformation. (We've seen a proof that the translation operator  $T(\varepsilon)$  (a special case of  $U(\varepsilon)$ ) is unitary and that its generator is Hermitian earlier, and the current case follows the same reasoning.) Using this form we have from 10 and 11, to order  $\varepsilon$ :

$$\delta X = \left( I + \frac{i\varepsilon}{\hbar} G \right) X \left( I - \frac{i\varepsilon}{\hbar} G \right) - X \quad (13)$$

$$= -\frac{i\varepsilon}{\hbar} [X, G] \quad (14)$$

$$\delta P = \left( I + \frac{i\varepsilon}{\hbar} G \right) P \left( I - \frac{i\varepsilon}{\hbar} G \right) - P \quad (15)$$

$$= -\frac{i\varepsilon}{\hbar} [P, G] \quad (16)$$

If  $G = P$ , then

$$\delta X = -\frac{i\varepsilon}{\hbar} [X, P] = \varepsilon I \quad (17)$$

$$\delta P = -\frac{i\varepsilon}{\hbar} [P, P] = 0 \quad (18)$$

Comparing this with 6 and 7 we see that (in one dimension, where the classical coordinate is given by  $x$  and momentum by  $p$ ) there is a correspondence between the classical Poisson bracket and quantum commutator:

$$\{x, p\} \leftrightarrow -\frac{i}{\hbar} [X, P] \quad (19)$$

The momentum operator  $P$  in quantum mechanics is thus the generator of translations, just as  $p$  generates translations in classical mechanics.

More generally, we can define the variation in some arbitrary dynamical operator  $\Omega$  in a similar way, using 12 to expand the RHS:

$$\delta\Omega = U^\dagger(\varepsilon)\Omega U(\varepsilon) - \Omega \quad (20)$$

$$= -\frac{i\varepsilon}{\hbar} [\Omega, G] \quad (21)$$

The correspondence with classical mechanics is then

$$\{\omega, g\} \leftrightarrow -\frac{i}{\hbar} [\Omega, G] \quad (22)$$

The general rule is that a quantum commutator is  $i\hbar$  times the corresponding classical Poisson bracket.

If  $\Omega = H$  and  $[H, G] = 0$ , then by Ehrenfest's theorem,  $\langle \dot{G} \rangle = 0$  and the average value of  $G$  is conserved.

The correspondence is a bit odd in that the generator  $g$  in classical mechanics enters as a derivative in 1 and 2 while the generator  $G$  in quantum mechanics enters as an operator (no derivatives) in 12.

One other feature is worth noting. A canonical transformation preserves the Poisson brackets 8 in the new coordinate system. In quantum mechanics, it is the commutators that get preserved. For example, using the fact that  $U$  is unitary so that  $UU^\dagger = I$ :

$$U^\dagger [X, P] U = U^\dagger X P U - U^\dagger P X U \quad (23)$$

$$= U^\dagger X U U^\dagger P U - U^\dagger P U U^\dagger X U \quad (24)$$

$$= [U^\dagger X U, U^\dagger P U] \quad (25)$$

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