TRANSLATION OPERATOR FROM PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11.

We've seen that the translation operator $T(\varepsilon)$ in quantum mechanics can be derived by considering the translation to be an active transformation, that is, a transformation where the state vectors, rather than the operators, get transformed according to

$$T(\varepsilon)|\psi\rangle=|\psi_{\varepsilon}\rangle \tag{0.1}$$

Using this approach, we found that

$$T(\varepsilon)=I-\frac{i\varepsilon}{\hbar}P \tag{0.2}$$

so that the momentum P is the generator of the transformation.

We can also derive T using a passive transformation, where the state vectors remain the same but the operators are transformed according to

$$T^{\dagger}(\varepsilon)XT(\varepsilon) = X+\varepsilon I \tag{0.3}$$

$$T^{\dagger}(\varepsilon)PT(\varepsilon) = P \tag{0.4}$$

This is equivalent to an active transformation since

$$\left\langle \psi\left|T^{\dagger}(\varepsilon)XT(\varepsilon)\right|\psi\right\rangle = \langle T(\varepsilon)\psi|X|T(\varepsilon)\psi\rangle \tag{0.5}$$

$$= \langle\psi_{\varepsilon}|X|\psi_{\varepsilon}\rangle \tag{0.6}$$

$$= x+\varepsilon \tag{0.7}$$

As before we start by taking

$$T(\varepsilon)=I-\frac{i\varepsilon}{\hbar}G \tag{0.8}$$

where G is some Hermitian operator, so that $G^{\dagger}=G$. Plugging this into 0.3 we get, keeping only terms up to order ε:

$$T^{\dagger}(\varepsilon)XT(\varepsilon) = \left(I+\frac{i\varepsilon}{\hbar}G\right)X\left(I-\frac{i\varepsilon}{\hbar}G\right) \tag{0.9}$$

$$= X+\frac{i\varepsilon}{\hbar}I(GX-XG) \tag{0.10}$$

$$= X-\frac{i\varepsilon}{\hbar}[X,G] \tag{0.11}$$

$$= X+\varepsilon I \tag{0.12}$$

Therefore

$$-\frac{i\varepsilon}{\hbar}[X,G] = \varepsilon I \tag{0.13}$$

$$[X,G] = i\hbar I \tag{0.14}$$

Since $[X,P]=i\hbar$ we see that

$$G = P+f(X) \tag{0.15}$$

The extra $f(X)$ is there because any function of X alone commutes with X, so

$$[X,G] = [X,P]+[X,f(X)] = i\hbar I+0 \tag{0.16}$$

We can eliminate $f(X)$ by considering 0.4.

$$T^{\dagger}(\varepsilon)PT(\varepsilon) = \left(I+\frac{i\varepsilon}{\hbar}G\right)P\left(I-\frac{i\varepsilon}{\hbar}G\right) \tag{0.17}$$

$$= P+\frac{i\varepsilon}{\hbar}I(GP-PG) \tag{0.18}$$

$$= P-\frac{i\varepsilon}{\hbar}[P,G] \tag{0.19}$$

$$= P \tag{0.20}$$

Thus we must have $[P,G]=0$, which means that G must be a function of P alone. This means that the most general form for $f(X)$ is $f(X)=$ constant, but there's nothing to be gained by adding some non-zero constant to G, so we can take $f(X)=0$. Thus we end up with the same form 0.2 that we got from the active transformation.

Translational invariance is the condition that the Hamiltonian is unaltered by a translation. In the passive representation this is stated by the condition

$$T^{\dagger}(\varepsilon)HT(\varepsilon)=H \tag{0.21}$$

Since translation is unitary, we can apply a theorem that is valid for any operator Ω which can be expanded in powers of X and P. For any unitary operator U, we have

$$U^{\dagger}\Omega(X,P)U=\Omega\left(U^{\dagger}XU,U^{\dagger}PU\right) \tag{0.22}$$

This follows because for a unitary operator $U^{\dagger}U=UU^{\dagger}=I$ so we can insert the product $UU^{\dagger}$ anywhere we like. In particular, we can insert it between each pair of factors in every term of the power series expansion of Ω, for example

$$U^{\dagger}X^{2}P^{2}U=U^{\dagger}XXPPU \tag{0.23}$$

$$=U^{\dagger}XUU^{\dagger}XUU^{\dagger}PUU^{\dagger}PU \tag{0.24}$$

$$=\left(U^{\dagger}XU\right)^{2}\left(U^{\dagger}PU\right)^{2} \tag{0.25}$$

For 0.21 this means that

$$T^{\dagger}(\varepsilon)H(X,P)T(\varepsilon)=H(X+\varepsilon I,P)=H(X,P) \tag{0.26}$$

As before, this leads to the condition

$$[P,H]=0 \tag{0.27}$$

which means that P is conserved, according to Ehrenfest's theorem.

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