TRANSLATION OPERATOR FROM PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11.

We've seen that the translation operator $T(\varepsilon)$ in quantum mechanics can be derived by considering the translation to be an active transformation, that is, a transformation where the state vectors, rather than the operators, get transformed according to

$$T(\varepsilon)|\psi\rangle=|\psi_{\varepsilon}\rangle \tag{1}$$

Using this approach, we found that

$$T(\varepsilon)=I-\frac{i\varepsilon}{\hbar}P \tag{2}$$

so that the momentum P is the generator of the transformation.

We can also derive T using a passive transformation, where the state vectors remain the same but the operators are transformed according to

$$\begin{aligned} T^{\dagger}(\varepsilon)XT(\varepsilon) &= X+\varepsilon I && (3)\\ T^{\dagger}(\varepsilon)PT(\varepsilon) &= P && (4) \end{aligned}$$

This is equivalent to an active transformation since

$$\begin{aligned} \left\langle\psi\left|T^{\dagger}(\varepsilon)XT(\varepsilon)\right|\psi\right\rangle &= \langle T(\varepsilon)\psi|X|T(\varepsilon)\psi\rangle && (5)\\ &= \langle\psi_{\varepsilon}|X|\psi_{\varepsilon}\rangle && (6)\\ &= x+\varepsilon && (7) \end{aligned}$$

As before we start by taking

$$T(\varepsilon)=I-\frac{i\varepsilon}{\hbar}G \tag{8}$$

where G is some Hermitian operator, so that $G^{\dagger}=G$. Plugging this into 3 we get, keeping only terms up to order ε:

$$T^{\dagger}(\varepsilon)XT(\varepsilon) = \left(I+\frac{i\varepsilon}{\hbar}G\right)X\left(I-\frac{i\varepsilon}{\hbar}G\right) \tag{9}$$
$$= X+\frac{i\varepsilon}{\hbar}I(GX-XG) \tag{10}$$
$$= X-\frac{i\varepsilon}{\hbar}[X,G] \tag{11}$$
$$= X+\varepsilon I \tag{12}$$

Therefore

$$-\frac{i\varepsilon}{\hbar}[X,G] = \varepsilon I \tag{13}$$
$$[X,G] = i\hbar I \tag{14}$$

Since $[X,P]=i\hbar$ we see that

$$G = P+f(X) \tag{15}$$

The extra $f(X)$ is there because any function of X alone commutes with X, so

$$[X,G]=[X,P]+[X,f(X)]=i\hbar I+0 \tag{16}$$

We can eliminate $f(X)$ by considering 4.

$$T^{\dagger}(\varepsilon)PT(\varepsilon) = \left(I+\frac{i\varepsilon}{\hbar}G\right)P\left(I-\frac{i\varepsilon}{\hbar}G\right) \tag{17}$$
$$= P+\frac{i\varepsilon}{\hbar}I(GP-PG) \tag{18}$$
$$= P-\frac{i\varepsilon}{\hbar}[P,G] \tag{19}$$
$$= P \tag{20}$$

Thus we must have $[P,G]=0$, which means that G must be a function of P alone. This means that the most general form for $f(X)$ is $f(X)=$ constant, but there's nothing to be gained by adding some non-zero constant to G, so we can take $f(X)=0$. Thus we end up with the same form 2 that we got from the active transformation.

Translational invariance is the condition that the Hamiltonian is unaltered by a translation. In the passive representation this is stated by the condition

$$T^{\dagger}(\varepsilon)HT(\varepsilon)=H \tag{21}$$

Since translation is unitary, we can apply a theorem that is valid for any operator Ω which can be expanded in powers of X and P. For any unitary operator U, we have

$$U^{\dagger}\Omega(X,P)U=\Omega\left(U^{\dagger}XU,U^{\dagger}PU\right) \tag{22}$$

This follows because for a unitary operator $U^{\dagger}U=UU^{\dagger}=I$ so we can insert the product $UU^{\dagger}$ anywhere we like. In particular, we can insert it between each pair of factors in every term of the power series expansion of Ω, for example

$$U^{\dagger}X^{2}P^{2}U=U^{\dagger}XXPPU \tag{23}$$
$$=U^{\dagger}XUU^{\dagger}XUU^{\dagger}PUU^{\dagger}PU \tag{24}$$
$$=\left(U^{\dagger}XU\right)^{2}\left(U^{\dagger}PU\right)^{2} \tag{25}$$

For 21 this means that

$$T^{\dagger}(\varepsilon)H(X,P)T(\varepsilon)=H(X+\varepsilon I,P)=H(X,P) \tag{26}$$

As before, this leads to the condition

$$[P,H]=0 \tag{27}$$

which means that P is conserved, according to Ehrenfest's theorem.

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