

## TRANSLATIONAL INVARIANCE AND CONSERVATION OF MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11.

One consequence of the invariance of the Hamiltonian under translation is that the momentum and Hamiltonian commute:

$$(0.1) \quad [P, H] = 0$$

In quantum mechanics, commuting quantities are simultaneously observable, and we can find a basis for the Hilbert space consisting of eigenstates of both  $P$  and  $H$ . We've seen that Ehrenfest's theorem allows us to conclude that for such a system, the average momentum is conserved so that  $\langle \dot{P} \rangle = 0$ . We can go a step further and state that if a system starts out in an eigenstate of  $P$ , then it remains in that eigenstate for all time.

First, we need to make a rather subtle observation, which is that

$$(0.2) \quad [P, H] = 0 \rightarrow [P, U(t)] = 0$$

That is, if  $P$  and  $H$  commute, then  $P$  also commutes with the propagator  $U(t)$ . For a time-independent Hamiltonian, the propagator is

$$(0.3) \quad U(t) = e^{-iHt/\hbar}$$

Since this can be expanded in a power series in the Hamiltonian, condition 0.2 follows easily enough. What if the Hamiltonian is time-dependent? In this case, the propagator comes out to a time-ordered integral

$$(0.4) \quad U(t) = T \left\{ \exp \left[ -\frac{i}{\hbar} \int_0^t H(t') dt' \right] \right\} \equiv \lim_{N \rightarrow \infty} \prod_{n=0}^{N-1} e^{-i\Delta H(n\Delta)/\hbar}$$

Here the time interval  $[0, t]$  is divided into  $N$  time slices, each of length  $\Delta = t/N$ . As explained in the earlier post, the reason we can't just integrate the RHS directly by summing the exponents is that such a procedure works only if the operators in the exponents all commute with each other. If  $H$  is time-dependent, its forms at different times may not commute, so we can't get a simple closed form for  $U(t)$ .

However, if  $[P, H(t)] = 0$  for all times, then  $P$  commutes with all the exponents on the RHS of 0.4, so we still get  $[P, U(t)] = 0$ . Another way of looking at this is by imposing the condition  $[P, H(t)] = 0$  we're saying that if  $H(t)$  can be expanded in a power series in  $X$  and  $P$ , it depends only on  $P$ , and not on  $X$ . This follows from the fact that

$$(0.5) \quad [X^n, P] = i\hbar n X^{n-1}$$

so that  $P$  does not commute with any power of  $X$ .

Given that 0.2 is valid for all Hamiltonians, then if we start in an eigenstate  $|p\rangle$  of  $P$ , then

$$(0.6) \quad P|p\rangle = p|p\rangle$$

$$(0.7) \quad PU(t)|p\rangle = U(t)P|p\rangle$$

$$(0.8) \quad = U(t)p|p\rangle$$

$$(0.9) \quad = pU(t)|p\rangle$$

Thus  $U(t)|p\rangle$  remains an eigenstate of  $P$  with the same eigenvalue  $p$  for all time. For a single particle moving in one dimension, the state  $|p\rangle$  describes a free particle with momentum  $p$  (and thus a completely undetermined position).

## PINGBACKS

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