FINITE TRANSFORMATIONS: CORRESPONDENCE BETWEEN CLASSICAL AND QUANTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11, Exercise 11.2.3.

The translation operator for an infinitesimal translation ε is, to first order in ε:

$$T(\varepsilon) = I - \frac{i\varepsilon}{\hbar}P \tag{1}$$

where P, the momentum operator, serves as the generator of translations. To derive a formula for a finite (non-infinitesimal) translation over a distance a, we divide the interval a into N segments, each of width a/N, so that for very large N, the width becomes infinitesimal. Then we have

$$T(a) = \left(I - \frac{ia}{\hbar N}P\right)^N \tag{2}$$

This formula is reminiscent of one definition of the exponential function (which can be found in most introductory calculus texts):

$$e^{-ax} = \lim_{N\to\infty}\left(1 - \frac{ax}{N}\right)^N \tag{3}$$

When we try to apply a formula that is valid for ordinary numbers to a case containing operators, we need to take care that any commutation relations involving the operators are taken into account. In this case, 2 contains only the momentum operator and the identity operator, which commute with each other, so we can in fact apply the limit formula directly to the operator case. We therefore have

$$T(a) = \lim_{N\to\infty}\left(I - \frac{ia}{\hbar N}P\right)^N = e^{-iaP/\hbar} \tag{4}$$

In the position basis, $P = -i\hbar\frac{d}{dx}$, so if we apply $T(a)$ to a state vector $\psi(x) = \langle x|\psi\rangle$ we can expand the exponential in a Taylor series to get

$$\langle x|T(a)|\psi\rangle = \psi(x) - a\frac{d\psi}{dx} + \frac{a}{2!}\frac{d^2\psi}{dx^2} + \ldots \tag{5}$$

We can extend our analysis of the correspondence between classical and quantum versions of translations. In the passive transformation model, the transformation is applied to operators rather than state vectors, so for a finite translation of an operator Ω we have

$$\Omega \to T^{\dagger}(a)\,\Omega T(a) = e^{iaP/\hbar}\Omega e^{-iaP/\hbar} \tag{6}$$

The operator expression on the RHS can be expanded using Hadamard's lemma, which for two operators A and B is

$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \ldots \tag{7}$$

where each term contains the commutator of the previous term's commutator with A.

In this case gives us

$$e^{iaP/\hbar}\Omega e^{-iaP/\hbar} = \Omega + a\left(-\frac{i}{\hbar}\right)[\Omega,P] + \frac{a^2}{2!}\left(-\frac{i}{\hbar}\right)^2[[\Omega,P],P] + \ldots \tag{8}$$

For example, in the case $\Omega = X$, $[X,P] = i\hbar I$ and all higher commutators are zero (since they involve the commutator of a constant with P), so we get

$$e^{iaP/\hbar}Xe^{-iaP/\hbar} = X + aI \tag{9}$$

so the system is translated by a distance a, as we'd expect.

For higher powers of X, we can use the result

$$[X^n,P] = i\hbar nX^{n-1} \tag{10}$$

We therefore get

$$\begin{aligned} e^{iaP/\hbar}X^n e^{-iaP/\hbar} &= X^n + anX^{n-1} + \frac{a^2}{2!}n(n-1)X^{n-2} + \ldots + \frac{a^n}{n!}n! && (11)\\ &= \sum_{m=0}^{n}\binom{n}{m}X^{n-m}(aI)^m && (12)\\ &= (X+aI)^n && (13) \end{aligned}$$

We're allowed to treat X as an ordinary number in these equations since it is (apart from I), the only operator present so all terms commute.

In the classical case, the infinitesimal change $\delta\omega$ of a variable ω under an infinitesimal displacement δa generated by the momentum p is given by the Poisson bracket

$$\delta\omega = \delta a \{\omega, p\} \tag{14}$$

We can write this as a derivative:

$$\frac{d\omega}{da} = \{\omega, p\} \tag{15}$$

For a finite translation by an amount a, we can write the value of ω as a Taylor series relative to some starting point a_0 as

$$\omega(a_0 + a) = \omega(a_0) + a\frac{d\omega}{da} + \frac{a^2}{2!}\frac{d^2\omega}{da^2} + \ldots \tag{16}$$

where all derivatives are evaluated at $a = a_0$.

We can write all the derivatives in terms of Poisson brackets by using 15. For example

$$\frac{d^2\omega}{da^2} = \frac{d}{da}\left(\frac{d\omega}{da}\right) = \left\{\frac{d\omega}{da}, p\right\} = \{\{\omega, p\}, p\} \tag{17}$$

Thus the variable ω transforms according to

$$\omega(a_0 + a) = \omega + a\{\omega, p\} + \frac{a^2}{2!}\{\{\omega, p\}, p\} + \ldots \tag{18}$$

Comparing this with 8, we see that the two expressions match if we use the usual recipe for converting classical Poisson brackets to quantum commutators, namely $\{a, b\} = -\frac{i}{\hbar}[A, B]$.

Although we've worked this out for the special case of translations, the same principle can be used for other transformations. For example, the angular momentum about the z axis is

$$\ell_z = xp_y - yp_x \tag{19}$$

and serves as the generator of rotations about the z axis. Suppose we have a rotation through an angle θ and we want to see how the two coordinates x and y transform. The expansion 18 becomes

$$\bar{x} = x + \theta\{x, \ell_z\} + \frac{\theta^2}{2!}\{\{x, \ell_z\}, \ell_z\} + \ldots \tag{20}$$

The relevant Poisson brackets are (using the generic term q_i to represent the two coordinates x and y):

$$\{x,\ell_z\} = \sum_i \left[\frac{\partial x}{\partial q_i}\frac{\partial \ell_z}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial \ell_z}{\partial q_i}\right] \tag{21}$$

$$= -y \tag{22}$$

$$\{y,\ell_z\} = \sum_i \left[\frac{\partial y}{\partial q_i}\frac{\partial \ell_z}{\partial p_i} - \frac{\partial y}{\partial p_i}\frac{\partial \ell_z}{\partial q_i}\right] \tag{23}$$

$$= x \tag{24}$$

Looking at how x transforms, we see that the Poisson brackets in 20 will cycle through the four values

$$\{x,\ell_z\} = -y \tag{25}$$

$$\{\{x,\ell_z\},\ell_z\} = -\{y,\ell_z\} = -x \tag{26}$$

$$\{\{\{x,\ell_z\},\ell_z\},\ell_z\} = -\{x,\ell_z\} = y \tag{27}$$

$$\{\{\{\{x,\ell_z\},\ell_z\},\ell_z\},\ell_z\} = \{y,\ell_z\} = x \tag{28}$$

The series 20 thus expands to

$$\bar{x} = x\left[1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right] - y\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right] \tag{29}$$

$$= x\cos\theta - y\sin\theta \tag{30}$$

We can do the same calculation for $\bar{y}$ to get

$$\bar{y} = x\sin\theta + y\cos\theta \tag{31}$$

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