

## TIME REVERSAL, ANTIUNITARY OPERATORS AND WIGNER'S THEOREM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11, Section 11.5.

Zee, A. (2016), *Group Theory in a Nutshell for Physicists*. Section IV.6.

Parity is one of the two main discrete symmetries treated in non-relativistic quantum mechanics. The other is time reversal, which we'll look at here.

First, we'll have a look at what time reversal symmetry means in classical physics. The idea is that if we can take a snapshot of the system at some time, each particle will have a given position  $x$  and a given momentum  $p$ . If we reverse the direction of time at that instant, the particle's position remains the same, but its momentum reverses. In other words  $x \rightarrow x$  and  $p \rightarrow -p$ . Note the difference between time reversal and parity: in a parity operation, *both* position and momentum get 'reflected' into their negative values, while in time reversal, only momentum gets 'reflected'.

We can see how this works by looking at Newton's law in the form

$$(0.1) \quad F = m \frac{d^2x}{dt^2}$$

Time reversal invariance is valid if the same equation holds when we reverse the direction of time, that is, we let  $t \rightarrow -t$ . Since  $x \rightarrow x$ , the numerator on the RHS is unchanged. For the denominator  $t \rightarrow -t$  means that  $dt \rightarrow -dt$  and  $(dt)^2 \rightarrow (-dt)^2 = dt^2$ , so the acceleration is invariant. Newton's law is invariant under time reversal provided that the force on the LHS is invariant, which will be the case provided that  $F$  depends only on  $x$  and not on  $\dot{x}$ . This is true for forces such as Newtonian gravity and electrostatics, but is not true for the magnetic force felt by a charge  $q$  moving through a magnetic field  $\mathbf{B}$  with velocity  $\mathbf{v}$ , where the Lorentz force law holds:

$$(0.2) \quad \mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

This follows because  $\mathbf{v} \rightarrow -\mathbf{v}$  so if the field  $\mathbf{B}$  is the same after time reversal,  $\mathbf{F} \rightarrow -\mathbf{F}$ . However, because all magnetic fields are produced by the motion of charges, if we expand the time reversal to include the charges giving rise to the magnetic field  $\mathbf{B}$ , then the motion of all these charges would

reverse, which in turn would cause  $\mathbf{B} \rightarrow -\mathbf{B}$ . Thus if we time-reverse the *entire* electromagnetic system, the electromagnetic force is invariant under time reversal.

How does time reversal work in quantum mechanics? Shankar considers a particle in one dimension governed by a time-independent Hamiltonian, which obeys the Schrödinger equation, as usual:

$$(0.3) \quad i\hbar \frac{\partial \psi(x,t)}{\partial t} = H(x) \psi(x,t)$$

At this point, Shankar states that if we replace  $\psi$  by its complex conjugate  $\psi^*$ , we are implementing time reversal, claiming that it is 'clear' because  $\psi^*$  gives the same probability distribution as  $\psi$ . I cannot find any reason why this should be 'clear' from this statement, so let's try looking at the problem in a bit more detail. The clearest explanation I've found is in Zee's book, referenced above.

In order that the system be invariant under time reversal, we consider the transformation  $t \rightarrow t' = -t$  and we wish to find some operator  $T$  which operates on the wave function  $\psi(t)$  so that

$$(0.4) \quad T\psi(t) = \psi'(t') = \psi'(-t)$$

[I'm suppressing the dependence on  $x$  for brevity; since time reversal doesn't affect  $x$ , it stays the same throughout this argument] satisfies the Schrödinger equation in the form

$$(0.5) \quad i\hbar \frac{\partial \psi'(t')}{\partial t'} = H\psi'(t')$$

From this, we get

$$(0.6) \quad i\hbar \frac{\partial (T\psi(t))}{\partial (-t)} = HT\psi(t)$$

Whatever this unknown operator  $T$  is, it has an inverse, so we can multiply on the left by  $T^{-1}$  to get

$$(0.7) \quad T^{-1}(-i)T\hbar \frac{\partial \psi(t)}{\partial t} = T^{-1}HT\psi(t)$$

Notice that we're not assuming that  $T$  has no effect on  $i$  (that is, we're not assuming that we can pull  $i$  out of the expression on the LHS). Now we know that  $T$  has an effect only if what it operates on depends on time (since it's the time reversal operator) so, since we're assuming that  $H$  is time-independent, we must have  $[H, T] = 0$ . Given this, we have

$$(0.8) \quad T^{-1}HT = T^{-1}TH = H$$

Thus, the RHS of 0.7 reduces to the RHS of the original Schrödinger equation 0.3. If the Schrödinger equation is to remain valid after time reversal, the LHS of 0.7 must also reduce to the LHS of 0.3. That is, we must have

$$(0.9) \quad T^{-1}(-i)T = i$$

Multiplying on the left by  $T$  we get

$$(0.10) \quad -iT = Ti$$

In other words, one of the effects of  $T$  is that it takes the complex conjugate of any expression that it operates on.

To find out exactly what  $T$  is, we can write it as the product of a unitary operator  $U$  and the operator  $K$ , whose only job is that it takes the complex conjugate. Since doing the complex conjugate operation twice in succession returns us to the original expression,  $K^2 = I$ , so  $K = K^{-1}$ . We get

$$(0.11) \quad T = UK$$

$$(0.12) \quad T^{-1} = K^{-1}U^{-1} = KU^{-1}$$

Ordinary unitary operators are linear in the sense that  $U(\alpha\psi) = \alpha U\psi$ , where  $\alpha$  is a complex number and  $\psi$  is some function, with a similar relation holding for  $U^{-1}$ . Combining the above few equations, we have

$$(0.13) \quad T^{-1}(-i)T = KU^{-1}(-i)UK$$

$$(0.14) \quad = K(-i)U^{-1}UK$$

$$(0.15) \quad = iK^2$$

$$(0.16) \quad = i$$

Thus the most general form for  $T$  is some unitary operator  $U$  multiplied by the complex conjugate operator  $K$ . We can see that, for such an operator, and complex constants  $\alpha$  and  $\beta$  and functions  $\psi$  and  $\phi$ :

$$(0.17) \quad T(\alpha\psi + \beta\phi) = UK(\alpha\psi + \beta\phi)$$

$$(0.18) \quad = U(\alpha^*K\psi + \beta^*K\phi)$$

$$(0.19) \quad = \alpha^*UK\psi + \beta^*UK\phi$$

$$(0.20) \quad = \alpha^*T\psi + \beta^*T\phi$$

An operator that obeys this relation is called *antilinear*. The operator  $T$  has the additional property

$$(0.21) \quad \langle T\psi | T\phi \rangle = \langle UK\psi | UK\phi \rangle$$

$$(0.22) \quad = \langle U\psi | U\phi \rangle^*$$

$$(0.23) \quad = \langle \psi | \phi \rangle^*$$

$$(0.24) \quad = \langle \phi | \psi \rangle$$

The third line follows from the fact that a unitary operator preserves inner products. An antilinear operator that satisfies the condition  $\langle T\psi | T\phi \rangle = \langle \phi | \psi \rangle$  is called *antiunitary*. [The fact that time reversal is antiunitary was first derived by Eugene Wigner in 1932. A more general result, known as Wigner's theorem, states that any symmetry in a quantum system must be represented by either a unitary or an antiunitary operator.]

To find  $U$  in this case, consider a plane wave state

$$(0.25) \quad \psi(t) = e^{i(px-Et)/\hbar}$$

Applying  $T$  to this state, we have

$$(0.26) \quad T\psi(t) = UKe^{i(px-Et)/\hbar}$$

$$(0.27) \quad = Ue^{-i(px-Et)/\hbar}$$

In one dimension, the only unitary operator  $U$  is a phase factor like  $e^{i\alpha}$  for some real  $\alpha$  (since  $U$  has to preserve the inner product). We can take  $U = 1$  since the phase factor cancels out when calculating  $|T\psi(t)|^2$ . Going back to 0.4, we see that the time-reversed wave function is

$$(0.28) \quad \psi'(-t) = T\psi(t) = e^{-i(px-Et)/\hbar}$$

$$(0.29) \quad \psi'(t) = e^{-i(px+Et)/\hbar} = e^{(-ipx-Et)/\hbar}$$

Since this is the same as the original wave function except that  $p \rightarrow -p$ , we see that it is indeed a valid time-reversed wave function. The energy

is the same (the  $-Et$  part of the exponent still has a minus sign) but the momentum has reversed, giving a wave that moves in the opposite direction.

Another way of looking at time reversal is as follows. Suppose we start with a system in the state  $\psi(0)$  at  $t = 0$ . We can let it evolve for a time  $\tau$  using the propagator to get the state at time  $t = \tau$ :

$$(0.30) \quad \psi(\tau) = e^{-iH\tau/\hbar} \psi(0)$$

Applying time reversal via the operator  $T$  to this state, we have (we're assuming that  $H$  is time-independent, but we're allowing it to be complex)

$$(0.31) \quad T\psi(\tau) = e^{iH^*\tau/\hbar} \psi^*(0)$$

If we now evolve this time-reversed state through the same time  $\tau$ , we should end up back in the (time-reversed) original state if the system is invariant under time reversal. That is,

$$(0.32) \quad \psi(2\tau) = e^{-iH\tau/\hbar} e^{iH^*\tau/\hbar} \psi^*(0) = \psi^*(0)$$

[Note that we don't require  $\psi(2\tau) = \psi(0)$  since  $\psi(2\tau)$  is the system in its time-reversed state, where it's moving in the opposite direction to the original state. Think about time-reversing a bouncing ball. The ball becomes effectively time-reversed when it bounces. If the ball is travelling down at some speed  $v$  at a height  $h$ , then after bouncing (assuming an elastic bounce) it will be travelling at the same speed  $v$  when it bounces back to the height  $h$ , but it will be moving in the opposite direction.]

In this equation, we're working in the  $X$  basis, so the exponents are numerical functions, not operators, and we're free to combine the exponents without worrying about commutators. This means that in order for the system to be time-reversal invariant, we must have

$$(0.33) \quad H(x) = H^*(x)$$

In other words, the Hamiltonian must be real. The usual kinetic plus potential type of Hamiltonian satisfies this since it has the form

$$(0.34) \quad H = \frac{P^2}{2m} + V(x)$$

and although the quantum momentum operator is  $P = -i\hbar \frac{d}{dx}$ , its square is real. In the magnetic force case, the presence of the charge's velocity as a linear term (in  $q\mathbf{v} \times \mathbf{B}$ ) means the momentum operator occurs as a linear term, making  $H$  complex, so time reversal invariance doesn't hold. Again,

however, if we included the charges that give rise to the magnetic field, the discrepancy disappears.