HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

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In the study of the harmonic oscillator, we can express $x$ and $p$ in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$  \hspace{1cm} (1)

$$p = i \sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-)$$  \hspace{1cm} (2)

We now have

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \int \psi_n^* (a_+ + a_-) \psi_n dx$$  \hspace{1cm} (3)

$$= 0$$  \hspace{1cm} (4)

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_+ \psi_n = \sqrt{n+1} \psi_{n+1}$$  \hspace{1cm} (5)

$$a_- \psi_n = \sqrt{n} \psi_{n-1}$$  \hspace{1cm} (6)

$$a_+ a_- \psi_n = n \psi_n$$  \hspace{1cm} (7)

$$a_- a_+ \psi_n = (n+1) \psi_n$$  \hspace{1cm} (8)

and since the wave functions are orthogonal, we get

$$\int \psi_n^* \psi_{n+1} dx = \int \psi_n^* \psi_{n-1} dx = 0$$  \hspace{1cm} (9)
Similarly:

\[ \langle p \rangle = i \sqrt{\frac{\bar{h} m \omega}{2}} \int \psi_n^* (a_+ - a_-) \psi_n dx \]  
\[ = 0 \]  
(10)  
(11)

for the same reason.

For the mean squares:

\[ \langle x^2 \rangle = \left( \frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ + a_-)(a_+ + a_-) \psi_n dx \]  
\[ = \left( \frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ a_- + a_- a_+) \psi_n dx \]  
\[ = \left( \frac{\hbar}{2m\omega} \right) (2n + 1) \]  
\[ = \frac{\hbar}{m\omega} \left( n + \frac{1}{2} \right) \]  
(12)  
(13)  
(14)  
(15)

In going from the first to the second line, we’ve thrown out terms where we integrate two orthogonal functions. For example,

\[ \int \psi_n^* a_+ a_+ \psi_n dx = \int \psi_n^* \sqrt{(n+1)(n+2)} \psi_{n+2} dx \]  
\[ = 0 \]  
(16)  
(17)

We have used the relations above and the fact that \( \psi_n \) is normalized to get the third line.

Similarly:

\[ \langle p^2 \rangle = -\frac{\hbar m \omega}{2} \int \psi_n^* (-a_+ a_- - a_- a_+) \psi_n dx \]  
\[ = \hbar m \omega \left( n + \frac{1}{2} \right) \]  
(18)  
(19)

The uncertainty principle then becomes

\[ \sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} \]  
\[ = \hbar \left( n + \frac{1}{2} \right) \]  
(20)  
(21)
and the kinetic energy is

\[ \langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{1}{2} \hbar \omega \left( n + \frac{1}{2} \right) \]  

(22)

which is half the total energy, as it should be.

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