Here are a couple of theorems that arise from the subspace theorem we proved earlier, which is:

If \( U \) is a subspace of \( V \), then \( V = U \oplus U^\perp \). (Recall the direct sum.)

Here, the orthogonal complement \( U^\perp \) of \( U \) is the set of all vectors that are orthogonal to all vectors \( u \in U \).

First, we can show that:

**Theorem 1.** The dimensionality of a vector space is \( n^\perp \), the maximum number of mutually orthogonal vectors in the space.

**Proof.** The set of mutually orthogonal vectors is linearly independent, and since it is the largest such set, any vector \( v \in V \) can be written as a linear combination of them. Thus the dimension of the space cannot be greater than \( n^\perp \). Since the set is linearly dependent, no member of the set can be written as a linear combination of the remaining members of the set, so the dimension can’t be less than \( n^\perp \). Thus the dimension must be equal to \( n^\perp \). \( \square \)

Now we look at a couple of other theorems.

**Theorem 2.** In a vector space \( V^n \) of dimension \( n \), the set \( V^\perp \) of all vectors orthogonal to any specific vector \( v \neq |0\rangle \) forms a subspace \( V^{n-1} \) of dimension \( n-1 \).

**Proof.** From the subspace theorem above, if we take \( U \) to be the subspace spanned by \( v \), then \( U^\perp \) is the orthogonal subspace. Since the dimension of \( U \) is 1 and \( V^n = U \oplus U^\perp \), the dimension of \( U^\perp = V^{n-1} \) is \( n-1 \). \( \square \)

**Theorem 3.** Given two subspaces \( V_1^{n_1} \) and \( V_2^{n_2} \) such that every vector \( v_1 \in V_1 \) is orthogonal to every vector \( v_2 \in V_2 \), the dimension of \( V_1 \oplus V_2 \) is \( n_1 + n_2 \).

**Proof.** An orthonormal basis of \( V_1 \) consists of \( n_1 \) mutually orthogonal vectors in \( V_1 \), and similarly, an orthonormal basis of \( V_2 \) consists of \( n_2 \) mutually orthogonal vectors in \( V_2 \). These bases consist of the maximum number of mutually orthogonal vectors in their respective spaces. In the direct sum \( V_1 \oplus V_2 \), we therefore have a set of \( n_1 + n_2 \) mutually orthogonal vectors,
which is the maximum number of such vectors in $V_1 \oplus V_2$. This follows because a vector $w \in V_1 \oplus V_2$ must be a linear combination of a vector $v_1 \in V_1$ and a vector $v_2 \in V_2$, where $v_i$ is, in turn, a linear combination of the basis of space $V_i$. Thus $w = v_1 + v_2$ must be a linear combination of vectors from the two bases combined. Hence the dimension of $V_1 \oplus V_2$ is $n_1 + n_2$. \qed