DETERMINANT AND TRACE OF NORMAL OPERATORS

The spectral theorem for states that any normal operator $\Omega$ in a complex vector space is unitarily diagonalizable, that is

$$D_M = U^\dagger \Omega U \quad (1)$$

where $U$ is a unitary operator and $D_M$ is a diagonal matrix, whose diagonal elements are the eigenvalues $\omega_i$ of $\Omega$. We can use this to derive a couple of relations about the trace and determinant of normal operators. Remember that hermitian and unitary operators are both normal.

Since the determinant is invariant under a unitary transformation, we have

$$\det D_M = \det \left( U^\dagger \Omega U \right) \quad (2)$$

$$= \det U^\dagger \det \Omega \det U \quad (3)$$

$$= e^{-i\alpha} \times \det \Omega \times e^{i\alpha} \quad (4)$$

$$= \det \Omega \quad (5)$$

where we’ve used the facts that the determinant of a product is the product of the determinants, and the determinant of a unitary matrix is a complex number $e^{i\alpha}$ with unit modulus. Since the determinant of a diagonal matrix is the product of its diagonal elements, we see that for a normal matrix, its determinant is the product of its eigenvalues:

$$\det \Omega = \prod_i \omega_i \quad (6)$$

The trace of a product is equal to the trace of a cyclic permutation of that product, so we have
\[
\text{Tr}D_M = \text{Tr} \left( U^\dagger \Omega U \right) \\
= \text{Tr} \left( U U^\dagger \Omega \right) \\
= \text{Tr}\Omega
\]

Therefore, the trace of a normal operator is the sum of its eigenvalues:

\[
\text{Tr}\Omega = \sum_i \omega_i
\]

We can use these two results as an alternative way to calculate the eigenvalues of a normal matrix. For example, suppose

\[
\Omega = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}
\]

We have

\[
\text{det}\Omega = -3 = \omega_1 \omega_2 \\
\text{Tr}\Omega = 2 = \omega_1 + \omega_2
\]

Solving these two equations gives

\[
-3 = (2 - \omega_2) \omega_2 \\
\omega = -1, 3
\]

We can also calculate them using the old determinant formula \(\text{det} (\Omega - \omega I) = 0:\)

\[
(1 - \omega)^2 - 4 = 0 \\
\omega = -1, 3
\]