EXPONENTIALS OF OPERATORS - HADAMARD’S LEMMA

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Although the result in this post isn’t covered in Shankar’s book, it’s a result that is frequently used in quantum theory, so it’s worth including at this point.
We’ve seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

\[ f(\Omega) = e^{i\Omega} \] (1)

Here we’ll look at a special function of two operators of the form

\[ h(A, B) = e^A Be^{-A} \] (2)

If \([A, B] = 0\), we can cancel the two exponentials and get the result \(h(A, B) = B\). However, if \([A, B] \neq 0\) the two exponentials must remain separated by the middle \(B\) operator. To get a simpler form for this function, we’ll consider the auxiliary function

\[ f(t) = e^{tA} Be^{-tA} \] (3)

where \(t\) is some parameter. We’ll need the first 3 derivatives at \(t = 0\):

\[
\begin{align*}
    f(0) &= B \\
    f'(t) &= Ae^{tA} Be^{-tA} - e^{tA} Be^{-tA}A \\
           &= e^{tA} [A, B] e^{-tA} \\
    f'(0) &= [A, B] \\
    f''(t) &= Ae^{tA} [A, B] e^{-tA} - e^{tA} [A, B] e^{-tA}A \\
           &= e^{tA} [A, [A, B]] e^{-tA} \\
    f''(0) &= [A, [A, B]] \\
    f'''(t) &= e^{tA} [A, [A, [A, B]]] e^{-tA} \\
    f'''(0) &= [A, [A, [A, B]]] 
\end{align*}
\]
We can now write a Taylor expansion of \( t^A e^{-tA} \) around \( t = 0 \):

\[
    e^tA Be^{-tA} = f(0) + tf'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(0) + \ldots
\]

(13)

\[
    \quad = B + [A, B] t + [A, [A, B]] \frac{t^2}{2!} + [A, [A, [A, B]]] \frac{t^3}{3!} + \ldots
\]

(14)

Taking \( t = 1 \) gives the required expansion

\[
    e^A Be^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \ldots
\]

(15)

This is known as Hadamard’s lemma.

If we introduce the notation

\[
    \text{ad}_A(B) \equiv [A, B]
\]

(16)

\[
    \text{ad}_A \text{ad}_A(B) = [A, [A, B]]
\]

(17)

and in general \((\text{ad}_A)^n(B)\) is the \( n \)th order commutator of \( A \) with \( B \), then we can write Hadamard’s lemma as

\[
    e^A Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_A)^n(B)
\]

(18)

\[
    = \exp(\text{ad}_A)(B)
\]

(19)

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