Continuing with our study of differential operators, we'll look now at their eigenvalues and eigenstates. The operator we're studying is

$$K = -i \frac{d}{dx} \quad (1)$$

The eigenvalue equation is as usual:

$$K |k\rangle = k |k\rangle \quad (2)$$

where $|k\rangle$ is an eigenstate and $k$ (outside the ket) is a (possibly complex) scalar. To find $|k\rangle$, we form the matrix element with $\langle x|$ and insert the unit operator:

$$\langle x |K| x\rangle = x k |k\rangle$$

$$\langle x |K| x\rangle = \int \langle x |K|x'\rangle \langle x'|k\rangle dx'$$

$$= -i \int \delta'(x-x') \psi_k(x') dx'$$

$$= -i \frac{d}{dx} \psi_k(x)$$

$$= -i \frac{d}{dx} \psi_k(x) \quad (6)$$

In the third line we used the matrix element

$$\langle x |K|x'\rangle = -i \delta'(x-x') \quad (7)$$

Equating the RHS on the first and last lines gives the differential equation

$$-i \frac{d}{dx} \psi_k(x) = k \psi_k(x) \quad (8)$$

which has the solution

$$\psi_k(x) = A e^{ikx} \quad (9)$$
where $A$ is a constant of integration. In order for $\psi_k(x)$ to be bounded as $x \to \pm \infty$, $k$ must be real, so we’ll restrict our attention to that case. The usual choice for $A$ is $1/\sqrt{2\pi}$ so that

$$
\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}
$$

This leads to the normalization condition

$$
\langle k | k' \rangle = \int_{-\infty}^{\infty} \langle k | x \rangle \langle x | k' \rangle \, dx
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k')x} \, dx
$$

$$
= \delta (k - k')
$$

where in the last line we used the traditional formula for the delta function. Thus the $|k\rangle$ basis is orthogonal, and normalized the same way as the $|x\rangle$ basis.

To convert between the $|k\rangle$ and $|x\rangle$ bases, we can use the unit operator in the two bases. Thus for some vector (function) $|f\rangle$ we have

$$
f(k) = \langle k | f \rangle = \int \langle k | x \rangle \langle x | f \rangle \, dx = \int \psi_k^*(x) f(x) \, dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x)
$$

(14)

Thus $f(k)$ is the Fourier transform of $f(x)$. We can use the same procedure to go in the reverse direction:

$$
f(x) = \langle x | f \rangle = \int \langle x | k \rangle \langle k | f \rangle \, dk = \int \psi_k(x) f(k) \, dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(k)
$$

(15)

The effect of the position operator $X$ on a vector $|f(x)\rangle$ can be found by inserting the unit operator:

$$
\langle x | X | f \rangle = \int \langle x | X | x' \rangle \langle x' | f \rangle \, dx'
$$

$$
= \int x' \langle x | x' \rangle \langle x' | f \rangle \, dx'
$$

$$
= \int x' \delta (x - x') \langle x' | f \rangle \, dx'
$$

$$
= x \langle x | f \rangle
$$

(19)

Thus $X$ just multiplies any function of $x$ by $x$ itself. A similar argument in the $|k\rangle$ basis shows that
\[ \langle k | K | f(k) \rangle = k \langle k | f(k) \rangle \]  
\hspace{1cm} (20)

We can use similar calculations to find the matrix elements of \( K \) in the \(|x\rangle\) basis and of \( X \) (the position operator) in the \(|k\rangle\) basis. We get

\[ \langle k | X | k' \rangle = \int \int \langle k | x \rangle \langle x | X | x' \rangle \langle x' | k' \rangle \, dx \, dx' \]  
\hspace{1cm} (21)
\[ = \frac{1}{2\pi} \int \int e^{-ikx} x' \langle x | x' \rangle e^{ik'x'} \, dx \, dx' \]  
\hspace{1cm} (22)
\[ = \frac{1}{2\pi} \int \int e^{-ikx} x' \delta(x - x') e^{ik'x'} \, dx \, dx' \]  
\hspace{1cm} (23)
\[ = \frac{1}{2\pi} \int xe^{i(k' - k)x} \, dx \]  
\hspace{1cm} (24)
\[ = i \frac{d}{dk} \left[ \frac{1}{2\pi} \int e^{i(k' - k)x} \, dx \right] \]  
\hspace{1cm} (25)
\[ = i \delta'(k - k') \]  
\hspace{1cm} (26)

The action of \( X \) on an arbitrary vector \(|g\rangle\) in the \(k\) basis can be found from this:

\[ \langle k | X | g(k) \rangle = \int \langle k | X | k' \rangle \langle k' | g \rangle \, dk' \]  
\hspace{1cm} (27)
\[ = i \int \delta'(k - k') g(k') \, dk' \]  
\hspace{1cm} (28)
\[ = i \frac{dg(k)}{dk} \]  
\hspace{1cm} (29)
\[ = i \left( k \left| \frac{dg(k)}{dk} \right. \right) \]  
\hspace{1cm} (30)

where in the third line we’ve used the property of \( \delta'(k - k') \) mentioned here.

By a similar calculation, we can find the matrix elements of \( K \) in the \(|x\rangle\) basis:
\[
\langle x | K | x' \rangle = \int \int \langle x | k \rangle \langle k | K | k' \rangle \langle k' | x' \rangle \, dk \, dk'
\]
\[
= \frac{1}{2\pi} \int \int e^{ikx} \langle k | k' \rangle e^{-ik'x'} \, dk \, dk'
\]
\[
= \frac{1}{2\pi} \int \int e^{ikx} k' \delta(k - k') e^{-ik'x'} \, dk \, dk'
\]
\[
= \frac{1}{2\pi} \int xe^{i(x-x')k} \, dk
\]
\[
= -i \frac{d}{dx} \left[ \frac{1}{2\pi} \int e^{i(x-x')k} \, dk \right]
\]
\[
= -i \delta'(x-x')
\]

Similarly, we have

\[
\langle x | K | g(x) \rangle = \int \langle x | K | x' \rangle \langle x' | g \rangle \, dx'
\]
\[
= -i \int \delta'(x-x') g(x') \, dx'
\]
\[
= -i \frac{dg(x)}{dx}
\]
\[
= -i \left\langle x \left| \frac{dg(x)}{dx} \right\rangle \right.
\]

From 30 and 40 we can work out the familiar commutator. Just for variety, we’ll do this in the \(|k\rangle\) basis:

\[
XK |f(k)\rangle = X |k |f(k)\rangle] = i \frac{d}{dk} \left[ |k |f(k)\rangle \right]
\]
\[
= i \left[ |f(k)\rangle + k \frac{df}{dk} \right]
\]
\[
KX |f(k)\rangle = iK \frac{df}{dk}
\]
\[
= ik \frac{df}{dk}
\]

Therefore

\[
[X, K] |f(k)\rangle = i |f(k)\rangle
\]
or, looking just at the operators

\[ [X, K] = iI \]  \hspace{1cm} (47)

PINGBACKS

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