LAGRANGIAN FOR A SPHERICALLY SYMMETRIC
POTENTIAL ENERGY FUNCTION

We now consider a more general example of the Euler-Lagrange equations of motion

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (1)
\]

where \( q_i \) and \( \dot{q}_i \) are the generalized coordinates and velocities, respectively. For systems where the potential energy \( V(q_i) \) is independent of the velocities \( \dot{q}_i \), the Lagrangian can be written as

\[
L = T - V \quad (2)
\]

where \( T \) is the kinetic energy.

Suppose we consider a system in three dimensions and use spherical coordinates to represent the position of a particle of mass \( m \). We’ll restrict ourselves to potential energy functions that depend only on the radial distance \( r \) from the origin, so that \( V(r, \theta, \phi) = V(r) \). To write down the Lagrangian, we need an expression for the kinetic energy \( T \).

An infinitesimal line element in spherical coordinates has a length \( ds \) given by

\[
\left( \frac{ds}{dt} \right)^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \quad (3)
\]

The square of the velocity is then given by dividing this expression through by \( dt^2 \), and using a dot above a symbol to indicate the derivative with respect to time \( t \). We have

\[
v^2 = \left( \frac{ds}{dt} \right)^2 = r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \quad (4)
\]

The Lagrangian is then
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\[ L = T - V \]

\[ = \frac{1}{2} m \left[ \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right] - V(r) \]

We now get three equations of motion by applying \[ I \] to each coordinate in turn. For \( r \):\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \]

\[ \ddot{r} = r \ddot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 - \frac{1}{m} \frac{dV}{dr} \]

For \( \theta \):\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \]

\[ \frac{d}{dt} (mr^2 \dot{\theta}) = mr^2 \sin \theta \cos \theta \dot{\phi}^2 \]

\[ 2r \ddot{r} + r^2 \ddot{\theta} = r^2 \sin \theta \cos \theta \dot{\phi}^2 \]

\[ \ddot{\theta} = -\frac{2}{r} \ddot{r} \dot{\theta} + \sin \theta \cos \theta \dot{\phi}^2 \]

For \( \phi \):\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \]

\[ \frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = 0 \]

\[ 2r \dot{r} \sin^2 \theta \dot{\phi} + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0 \]

\[ \ddot{\phi} = -\frac{2}{r} \ddot{r} \dot{\phi} - 2 \cot \theta \dot{\theta} \dot{\phi} \]

Although the only equation in which the potential energy \( V \) has a direct effect is the one for \( r \), these three equations constitute a system of non-linear coupled differential equations so in the general case, they can be difficult to solve.

One important special case is that of a path that lies in the plane \( \theta = \frac{\pi}{2} \), such as the orbit of a planet around the sun. In that case \( \ddot{\theta} = 0 \), \( \sin \theta = 1 \) and \( \cos \theta = 0 \), so the equations simplify to
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\[ \ddot{r} = r \dot{\phi}^2 - \frac{1}{m} \frac{dV}{dr} \] (17)

\[ \ddot{\theta} = 0 \] (18)

\[ \ddot{\phi} = -\frac{2}{r} \dot{r} \dot{\phi} \] (19)