CHANGING THE POSITION BASIS WITH A UNITARY TRANSFORMATION

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Section 7.4, Exercise 7.4.9.
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The standard representation of the position and momentum operators in the position basis is

\[ X \rightarrow x \] (1)
\[ P \rightarrow -i\hbar \frac{d}{dx} \] (2)

It turns out it’s possible to modify this definition by adding some arbitrary function of position \( f(x) \) to \( P \) so we have

\[ X' \rightarrow x \] (3)
\[ P' \rightarrow -i\hbar \frac{d}{dx} + f(x) \] (4)

Since any function of \( x \) commutes with \( X \), the commutation relations remain unchanged, so we have

\[ [X', P'] = i\hbar \] (5)

Another way of interpreting this change in operators is by using the unitary transformation of the \( X \) basis, in the form

\[ |x\rangle \rightarrow |\bar{x}\rangle = e^{ig(X)/\hbar} |x\rangle = e^{ig(x)/\hbar} |x\rangle \] (6)

where

\[ g(x) \equiv \int^x f(x') \, dx' \] (7)

The last equality in (6) comes from the fact that operating on \( |x\rangle \) with any function of the \( X \) operator (provided the function can be expanded in a power series) results in multiplying \( |x\rangle \) by the same function, but with the operator \( X \) replaced by the numeric position value.
To verify this works, we can calculate the matrix elements of the old $X$ and $P$ operators in the new basis. We have

$$\langle \tilde{x} | X | \tilde{x}' \rangle = \langle x | e^{-i g(x)/\hbar} X e^{i g(x')/\hbar} | x' \rangle$$

(8)

At this stage, since the two exponentials are numerical functions and not operators, we can take them outside the bracket to

$$\langle \tilde{x} | X | \tilde{x}' \rangle = e^{-i g(x)/\hbar} e^{i g(x')/\hbar} \langle x | X | x' \rangle$$

(9)

$$= e^{-i g(x)/\hbar} e^{i g(x')/\hbar} x' \delta (x - x')$$

(10)

The exponentials cancel in the last line since the delta function is non-zero only when $x = x'$.

The above result can also be obtained by inserting a couple of identity operators into $8$:

$$\langle \tilde{x} | X | \tilde{x}' \rangle = \int \int \langle x | e^{-i g(x)/\hbar} | y \rangle \langle y | X | z \rangle \langle z | e^{i g(x')/\hbar} | x' \rangle dy dz$$

(12)

$$= \int \int \langle x | e^{-i g(x)/\hbar} | y \rangle z \delta (y - z) \langle z | e^{i g(x')/\hbar} | x' \rangle dy dz$$

(13)

$$= \int \langle x | e^{-i g(x)/\hbar} | z \rangle z \langle z | e^{i g(x')/\hbar} | x' \rangle dz$$

(14)

$$= e^{i [g(x') - g(x)]/\hbar} \langle x | z \rangle z \langle z | x' \rangle dz$$

(15)

$$= e^{i [g(x') - g(x)]/\hbar} \delta (x - z) z \delta (z - x') dz$$

(16)

$$= e^{i [g(x') - g(x)]/\hbar} x' \delta (x - x')$$

(17)

$$= x \delta (x - x')$$

(18)

The momentum operator works as follows. Using the original definition on the modified basis we have
\[ \langle \tilde{x} | P | \tilde{x}' \rangle = -i\hbar \langle x \bigg| e^{-ig(x)/\hbar} \frac{d}{dx'} e^{ig(x')/\hbar} \bigg| x' \rangle \]  
(19)

\[ = -i\hbar \langle x \bigg| e^{-ig(x)/\hbar} i\frac{h}{\hbar} e^{ig(x')/\hbar} \frac{dg(x')}{dx'} \bigg| x' \rangle - \]  
(20)

\[ i\hbar \langle x \bigg| e^{-ig(x)/\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \bigg| x' \rangle \]  
(21)

From (7) we have

\[ \frac{dg(x)}{dx} = \frac{d}{dx} \int x f(x') dx' = f(x) \]  
(22)

This gives

\[ \langle \tilde{x} | P | \tilde{x}' \rangle = \langle x \bigg| e^{ig(x')-g(x)/\hbar} \left[ f(x') - i\hbar \frac{d}{dx'} \right] \bigg| x' \rangle \]  
(23)

\[ = \left[ f(x') - i\hbar \frac{d}{dx'} \right] \langle x | x' \rangle \]  
(24)

\[ = \left[ f(x') - i\hbar \frac{d}{dx'} \right] \delta(x - x') \]  
(25)

\[ = \left[ f(x) - i\hbar \frac{d}{dx} \right] \delta(x - x') \]  
(26)

This shows that by a unitary change of \(X\) basis (6) we transform the position and momentum operators (well, just the momentum operator, really) according to (3). We’ve multiplied the original \( |x\rangle \) states by a phase factor which depends on some function \( f(x) \). This doesn’t change the matrix elements of \(X\), but it does add \( f(x)\) to the matrix elements of \(P\). The commonly used definition of \(P\) is thus with \( f(x) = 0.\)