TRANSLATION OPERATOR FROM PASSIVE TRANSFORMATIONS

We’ve seen that the translation operator $T(\varepsilon)$ in quantum mechanics can be derived by considering the translation to be an active transformation, that is, a transformation where the state vectors, rather than the operators, get transformed according to

$$ T(\varepsilon) |\psi\rangle = |\psi_{\varepsilon}\rangle \quad (1) $$

Using this approach, we found that

$$ T(\varepsilon) = I - \frac{i \varepsilon}{\hbar} P \quad (2) $$

so that the momentum $P$ is the generator of the transformation.

We can also derive $T$ using a passive transformation, where the state vectors remain the same but the operators are transformed according to

$$ T^\dagger(\varepsilon) X T(\varepsilon) = X + \varepsilon I \quad (3) $$
$$ T^\dagger(\varepsilon) P T(\varepsilon) = P \quad (4) $$

This is equivalent to an active transformation since

$$ \langle \psi | T^\dagger(\varepsilon) X T(\varepsilon) | \psi \rangle = \langle T(\varepsilon) \psi | X | T(\varepsilon) \psi \rangle \quad (5) $$
$$ = \langle \psi_{\varepsilon} | X | \psi_{\varepsilon} \rangle \quad (6) $$
$$ = \langle \psi_{\varepsilon} | X | \psi_{\varepsilon} \rangle \quad (7) $$

As before we start by taking

$$ T(\varepsilon) = I - \frac{i \varepsilon}{\hbar} G \quad (8) $$

where $G$ is some Hermitian operator, so that $G^\dagger = G$. Plugging this into we get, keeping only terms up to order $\varepsilon$:
\[
T^\dagger (\varepsilon) X T (\varepsilon) = \left( I + \frac{i\varepsilon}{\hbar} G \right) X \left( I - \frac{i\varepsilon}{\hbar} G \right) \quad (9)
\]
\[
= X + \frac{i\varepsilon}{\hbar} I (GX - XG) \quad (10)
\]
\[
= X - \frac{i\varepsilon}{\hbar} [X, G] \quad (11)
\]
\[
= X + \varepsilon I \quad (12)
\]

Therefore

\[
-\frac{i\varepsilon}{\hbar} [X, G] \quad = \quad \varepsilon I \quad (13)
\]
\[
[X, G] \quad = \quad i\hbar I \quad (14)
\]

Since \([X, P] = \hbar\) we see that

\[G = P + f(X) \quad (15)\]

The extra \(f(X)\) is there because any function of \(X\) alone commutes with \(X\), so

\[ [X, G] = [X, P] + [X, f(X)] = i\hbar I + 0 \quad (16) \]

We can eliminate \(f(X)\) by considering \(4\).

\[
T^\dagger (\varepsilon) P T (\varepsilon) = \left( I + \frac{i\varepsilon}{\hbar} G \right) P \left( I - \frac{i\varepsilon}{\hbar} G \right) \quad (17)
\]
\[
= P + \frac{i\varepsilon}{\hbar} I (GP - PG) \quad (18)
\]
\[
= P - \frac{i\varepsilon}{\hbar} [P, G] \quad (19)
\]
\[
= P \quad (20)
\]

Thus we must have \([P, G] = 0\), which means that \(G\) must be a function of \(P\) alone. This means that the most general form for \(f(X)\) is \(f(X) = \) constant, but there’s nothing to be gained by adding some non-zero constant to \(G\), so we can take \(f(X) = 0\). Thus we end up with the same form \(2\) that we got from the active transformation.

Translational invariance is the condition that the Hamiltonian is unaltered by a translation. In the passive representation this is stated by the condition

\[
T^\dagger (\varepsilon) H T (\varepsilon) = H \quad (21)
\]
Since translation is unitary, we can apply a theorem that is valid for any operator $\Omega$ which can be expanded in powers of $X$ and $P$. For any unitary operator $U$, we have

$$U^\dagger \Omega(X, P) U = \Omega(U^\dagger XU, U^\dagger PU)$$  \hspace{1cm} (22)

This follows because for a unitary operator $U^\dagger U = UU^\dagger = I$ so we can insert the product $UU^\dagger$ anywhere we like. In particular, we can insert it between each pair of factors in every term of the power series expansion of $\Omega$, for example

$$U^\dagger X^2 P^2 U = U^\dagger XXP\, PP\, U$$
\hspace{1cm} (23)

$$= U^\dagger XUU^\dagger XUU^\dagger PUU^\dagger PU$$
\hspace{1cm} (24)

$$= \left(U^\dagger XU\right)^2 \left(U^\dagger PU\right)^2$$
\hspace{1cm} (25)

For $21$ this means that

$$T^\dagger (\varepsilon) H(X, P) T(\varepsilon) = H(X + \varepsilon I, P) = H(X, P)$$
\hspace{1cm} (26)

As before, this leads to the condition

$$[P, H] = 0$$
\hspace{1cm} (27)

which means that $P$ is conserved, according to Ehrenfest’s theorem.

**Pingbacks**

- Translational invariance and conservation of momentum
- Finite transformations: correspondence between classical and quantum
- Parity transformations
- Rotational transformations using passive transformations