TRANSLATIONAL INVARIANCE AND CONSERVATION OF MOMENTUM

One consequence of the invariance of the Hamiltonian under translation is that the momentum and Hamiltonian commute:

\[ [P, H] = 0 \] (1)

In quantum mechanics, commuting quantities are simultaneously observable, and we can find a basis for the Hilbert space consisting of eigenstates of both \( P \) and \( H \). We’ve seen that Ehrenfest’s theorem allows us to conclude that for such a system, the average momentum is conserved so that \( \langle \dot{P} \rangle = 0 \). We can go a step further and state that if a system starts out in an eigenstate of \( P \), then it remains in that eigenstate for all time.

First, we need to make a rather subtle observation, which is that \( [P, H] = 0 \rightarrow [P, U(t)] = 0 \) (2)

That is, if \( P \) and \( H \) commute, then \( P \) also commutes with the propagator \( U(t) \). For a time-independent Hamiltonian, the propagator is

\[ U(t) = e^{-iHt/\hbar} \] (3)

Since this can be expanded in a power series in the Hamiltonian, condition 2 follows easily enough. What if the Hamiltonian is time-dependent? In this case, the propagator comes out to a time-ordered integral

\[ U(t) = T \left\{ \exp \left[ -\frac{i}{\hbar} \int_0^t H(t') \, dt' \right] \right\} = \lim_{N \to \infty} \prod_{n=0}^{N-1} e^{-i\Delta H(n\Delta)/\hbar} \] (4)

Here the time interval \([0, t]\) is divided into \( N \) time slices, each of length \( \Delta = t/N \). As explained in the earlier post, the reason we can’t just integrate the RHS directly by summing the exponents is that such a procedure works only if the operators in the exponents all commute with each other. If \( H \) is time-dependent, its forms at different times may not commute, so we can’t get a simple closed form for \( U(t) \).
However, if \([P,H(t)] = 0\) for all times, then \(P\) commutes with all the exponents on the RHS of \([4]\), so we still get \([P,U(t)] = 0\). Another way of looking at this is by imposing the condition \([P,H(t)] = 0\) we’re saying that if \(H(t)\) can be expanded in a power series in \(X\) and \(P\), it depends only on \(P\), and not on \(X\). This follows from the fact that

\[ [X^n, P] = i\hbar n X^{n-1} \]  

so that \(P\) does not commute with any power of \(X\).

Given that \([2]\) is valid for all Hamiltonians, then if we start in an eigenstate \(|p\rangle\) of \(P\), then

\[
P|p\rangle = p|p\rangle \tag{6}
\]

\[
PU(t)|p\rangle = U(t)P|p\rangle = U(t)p|p\rangle \tag{7}
\]

\[
= pU(t)|p\rangle \tag{8}
\]

Thus \(U(t)|p\rangle\) remains an eigenstate of \(P\) with the same eigenvalue \(p\) for all time. For a single particle moving in one dimension, the state \(|p\rangle\) describes a free particle with momentum \(p\) (and thus a completely undetermined position).

Pingbacks

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